Math 463/563
Homework #5 - Due Wednesday, December 2

1. The density function of X is given by

$$f(x) = \begin{cases} \frac{c}{x^4} & x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that $c = 3$.
(ii) Compute $E[X]$ and $Var(X)$.

2. The standard normal random variable Z is characterized by its density function

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}.$$

Compute $E[e^Z]$.

3. We know that for any finite collection of random variables (not necessarily independent), the expectation of their sum is equal to the sum of their expectations. If X and Y are independent random variables, prove

$$Var(X+Y) = Var(X) + Var(Y)$$

Hint: Recall that if X and Y are independent random variables, then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

4. Let X_1, X_2, and X_3 each be uniform random variables over $[0,1]$. If in addition X_1, X_2, and X_3 are independent, find the probability density function $f_{x_1+x_2+x_3}$ of their sum $X_1 + X_2 + X_3$.

Hint: The density function $f_{x_1+x_2}$ of $X_1 + X_2$ was found in class:

$$f_{x_1+x_2}(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Next, notice that $f_{x_1+x_2+x_3}(a) = \int\limits_{-\infty}^{\infty} f_{x_1+x_2}(x) f_{x_3}(a-x)\, dx$

5. An experiment consists of $1,210$ independent Bernoulli trials with probability $p = \frac{1}{11}$ of success. Use the Central Limit Theorem for Bernoulli Trials of section 9.1 (the DeMoivre-Laplace theorem) and the table of values for the standard normal distribution to estimate the probability of the event that

$$\{98 \leq \text{ the number of successes } \leq 116\}.$$

Remember: it is best to consider $P\{97.5 \leq \text{ the number of successes } \leq 116.5\}$.

6. As part of the gambling game at the "*Magic * 463/563 * Casino*", a fair coin ($p = \frac{1}{2}$) is flipped $90,000$ times. Use the Central Limit Theorem for Bernoulli Trials of section 9.1 (the DeMoivre-Laplace theorem) and the table of values for the standard normal distribution to estimate the probability of the event that

$$\{45,032 \leq \text{ the number of heads } \leq 45,069\}.$$

7. A fair die is rolled 18,000 times. Estimate the probability that '6' comes up at least 3,060 times.