Math 463/563
Homework #4 - Due Monday, November 16

1. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{8}{x^3} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Check that $f(x)$ is indeed a probability density function. Find $P(X > 5)$ and $E[X]$.

2. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} c(x-1)^4 \text{ if } 1 < x < 2, \\ 0 \text{ otherwise} \end{cases}$$

wher c is a constant. Find c, and $E[X]$.

3. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} ax^2 + bx & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants. Suppose $E[X] = 0.75$. Find a, b, $E[X^2]$ and $Var(X)$.

4. Suppose the cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (x+1)^{-2} \text{ if } x > 0, \\ 0 \text{ if } x \leq 0. \end{cases}$$

Evaluate $P(1 < X < 3)$ and $E[X]$.

5. If Y is an exponential random variable with parameter $\lambda = 3$, what is the probability that the roots of the equation

$$4x^2 + 4xY - Y + 6 = 0$$

are real?

6. The *gamma function* $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

for all $\alpha > 0$. Use integration by parts to prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Compute $\Gamma(1)$ and show that $\Gamma(k) = (k-1)!$ for all positive integer k.

7. Use the preceding exercise to show that if X is an exponential random variable with $\lambda > 0$,

$$E[X^k] = \frac{k!}{\lambda^k}$$

for all positive integer $k = 1, 2, \ldots$.

8. A gamma distributed random variable with parameters (α, λ) is defined by its probability density function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} & \text{when } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. Suppose X is a gamma distributed random variable with parameters (α, λ), where $\alpha > 0$ and $\lambda > 0$. Compute $E[e^{-X}]$.

9. Let $f(t)$ be the probability density function, and $F(t)$ be the corresponding cumulative distribution function. Define the *hazard function* $h(t) = \frac{f(t)}{1-F(t)}$. Show that if X is an exponential random variable with parameter $\lambda > 0$, then its hazard function will be a constant

$$h(t) = \lambda$$

for all $t > 0$. Think of how this relates to the memorylessness property of exponential random variables.