Math 463/563
Homework #3 - Due Friday, October 30

1. Let random variable X have probability mass function

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 2 \\ \frac{1}{6} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

(i.e. $p_X(0) = \frac{1}{3}$, $p_X(2) = \frac{1}{2}$ and $p_X(3) = \frac{1}{6}$).

Compute $E[X]$, $Var(X)$, $E[|X - E[X]|]$, and $E[2^X]$.

2. Let X be a Poisson random variable with parameter $\lambda > 0$ (recall that X is Poisson with parameter λ if its probability mass function is given by $p_X(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$ for $i = 0, 1, 2, ...$). Find $E\left[\frac{1}{X+1}\right]$.

3. Let X be a binomial random variable with parameters (n, p). Show that

$$E\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

4. For a nonnegative integer-valued random variable X, show that

$$E[X] = \sum_{j=1}^{\infty} P(X \geq j)$$

Hint: Write $\sum_{j=1}^{\infty} P(X \geq j) = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} P(X = k)$, interchange the order of summation.

5. Recall that a *geometric* random variable with parameter $p \in (0, 1)$ is defined by its geometric probability mass function

$$p(i) = p(1-p)^{i-1} \text{ for } i = 1, 2, 3, ...$$

Suppose W is such random variable. Find $P\{W > j\}$ $(j = 0, 1, 2, ...)$. Use the identity $E[W] = \sum_{j=0}^{+\infty} P\{W > j\}$ from the previous problem to prove that $E[W] = \frac{1}{p}$.

6. Let X be a random variable such that $P(X = 1) = p$ and $P(X = -1) = 1 - p$. Find $a \neq 1$ such that
$$E[a^X] = 1$$

7. Consider a discrete random variable X. Use the definition of variance to prove that
$$E[X^2] \geq (E[X])^2$$

8. Consider a random variable X with $E[X] = \mu$ and $Var(X) = \sigma^2$. Let $Y = \frac{X-\mu}{\sigma}$. Find $E[Y]$ and $Var(Y)$.

9. A rocket engine fails with probability $1 - p$ independently from engine to engine. In order to launch a satellite into space one needs the majority of engines on a rocket to work. For what values of p would a 3-engine rocket be more reliable than a 5-engine rocket?