MTH 463/563

Lectures 23-28

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- Independent random variables.
- Sums of random variables. Examples.
- Law of large numbers (LLN).
- Central limit theorem (CLT).
- de Moivre's and Stirling's formulas.
- De Moivre - Laplace Theorem.

Independent random variables.

Problem. We say that two discrete random variables X and Y, are independent when

$$P\Big(\{X = a\} \cap \{Y = b\}\Big) = P(X = a)P(Y = b)$$

for all a and b in the corresponding sample spaces. Let X_1 X_1 and X_2 be independent Poisson random variables with parameters $\lambda_1 = 3$ and $\lambda_2 = 2$. Find $P(X_1 + X_2 = 3)$.

Solution: Since

$$\{X_1+X_2 = 3\} = \{X_1 = 0, X_2 = 3\}\cup\{X_1 = 1, X_2 = 2\}\cup\{X_1 = 2, X_2 = 1\}\cup\{X_1 = 3, X_2 = 0\}$$

is the union of mutually exclusive (disjoint) events,

$$P(X_1 + X_2 = 3) = P(X_1 = 0, X_2 = 3) + P(X_1 = 1, X_2 = 2)$$

$$+P(X_1 = 2, X_2 = 1)+P(X_1 = 3, X_2 = 0)$$

Independent random variables.

Solution (continued): Since

$$\{X_1+X_2=3\}=\{X_1=0,X_2=3\}\cup\{X_1=1,X_2=2\}\cup\{X_1=2,X_2=1\}\cup\{X_1=3,X_2=0\}$$

is the union of mutually exclusive (disjoint) events,

$$P(X_1+X_2=3)=P(X_1=0,X_2=3)+P(X_1=1,X_2=2)$$
$$+P(X_1=2,X_2=1)+P(X_1=3,X_2=0)$$

Since X_1 and X_2 are independent,

$$P(X_1+X_2=3)=P(X_1=0)P(X_2=3)+P(X_1=1)P(X_2=2)$$
$$+P(X_1=2)P(X_2=1)+P(X_1=3)P(X_2=0)$$

Plug in the probabilities to get

$$e^{-5}\left(1\cdot\frac{2^3}{3!}+\frac{3}{1!}\cdot\frac{2^2}{2!}+\frac{3^2}{2!}\cdot\frac{2}{1!}+\frac{3^3}{3!}\cdot 1\right)=e^{-5}(4/3+6+9+9/2)=e^{-5}\frac{125}{6}$$

Independent random variables.

Definition. Random variables X and Y are said to be **independent** if the events

$$\{a \le X \le b\} \quad \text{and} \quad \{c \le Y \le d\}$$

are independent, for any a, b, c, and d. Namely,

$$P\Big(\{a \le X \le b\} \cap \{c \le Y \le d\}\Big) = P(a \le X \le b)\, P(c \le Y \le d)$$

Similarly for n independent random variables.

Properties:

• If X and Y are both discrete random variables. They are independent if and only if

$$P\Big(\{X = a\} \cap \{Y = b\}\Big) = P(X = a)\, P(Y = b)$$

for all a and b in the corresponding sample spaces.

• If X and Y are independent then

$$E[f(X)g(Y)] = E[f(X)]\, E[g(Y)]$$

for any pair of functions, f and g.

Sums of independent random variables.

• If X and Y are independent discrete random variables, with probability mass functions $p_{\mathrm{x}}(x) = P(X = x)$ and $p_{\mathrm{y}}(y) = P(Y = y)$, then their sum, $Z = X + Y$ is also a discrete random variable with probability mass function

$$p_{\mathrm{z}}(a) \;=\; \sum_{x,y:\; x+y=a} p_{\mathrm{x}}(x)\, p_{\mathrm{y}}(y)$$

which can be rewritten as a **convolution sum**:

$$p_{\mathrm{z}}(a) \;=\; \sum_{x} p_{\mathrm{x}}(x)\, p_{\mathrm{y}}(a-x)$$

• If X and Y are independent continuous random variables, with density functions f_{x} and f_{y}, then $Z = X + Y$ is also a continuous random variable with its density f_Z given as a **convolution integral**,

$$f_{\mathrm{z}}(a) = \int_{-\infty}^{\infty} f_{\mathrm{x}}(x)\, f_{\mathrm{y}}(a-x)\; dx$$

Sums of independent random variables.

$$p_{x+y}(a) = \sum_x p_x(x)\, p_y(a-x)$$

• **Example.** Let X be binomial with parameters (n_1, p) and Y be binomial with parameters (n_2, p). Then their probability mass functions are

$$p_x(k) = \binom{n_1}{k} p^k (1-p)^{n_1-k} \qquad \text{for } k = 0, 1, \ldots, n_1$$

and

$$p_y(k) = \binom{n_2}{k} p^k (1-p)^{n_2-k} \qquad \text{for } k = 0, 1, \ldots, n_2$$

If X and Y are independent, then their sum will have the following distribution: for $j = 0, 1, \ldots, n_1 + n_2$,

$$p_{x+y}(j) = \sum_k p_x(k) p_y(j-k) = \sum_{\substack{0 \le k \le n_1 \\ j-n_2 \le k \le j}} \binom{n_1}{k} p^k (1-p)^{n_1-k} \binom{n_2}{j-k} p^{j-k} (1-p)^{n_2-j+k}$$

$$= p^j (1-p)^{n_1+n_2-j} \sum_{\substack{0 \le k \le n_1 \\ j-n_2 \le k \le j}} \binom{n_1}{k} \binom{n_2}{j-k}$$

Sums of independent random variables.

$$p_{X+Y}(a) \;=\; \sum_x p_X(x)\, p_Y(a-x)$$

• **Example (continued).** Let X be binomial with parameters (n_1, p) and Y be binomial with parameters (n_2, p). If X and Y are independent, then their sum will have the following distribution: for $j = 0, 1, \ldots, n_1 + n_2$,

$$p_{X+Y}(j) \;=\; \sum_k p_X(k)\, p_Y(j-k) \;=\; p^j\,(1-p)^{n_1+n_2-j} \sum_{\substack{0\le k\le n_1\\ j-n_2\le k\le j}} \binom{n_1}{k}\binom{n_2}{j-k},$$

where $\sum_{\substack{0\le k\le n_1\\ j-n_2\le k\le j}} \binom{n_1}{k}\binom{n_2}{j-k} = \binom{n_1+n_2}{j}$ since both sides represent the number of $n_1 + n_2$ long strings one can make with j A's and $n_1 + n_2 - j$ B's.

Hence,

$$p_{X+Y}(j) \;=\; \binom{n_1+n_2}{j}\, p^j\,(1-p)^{n_1+n_2-j} \qquad \text{for } j = 0, 1, \ldots, n_1 + n_2$$

Thus $X + Y$ is binomial with parameters $(n_1 + n_2, p)$.

Sums of independent random variables.

$$p_{x+y}(a) = \sum_x p_x(x)\, p_y(a-x)$$

• **Example.** Let X be Poisson with parameter λ_1 and Y be Poisson with parameter λ_2. Then their probability mass functions are

$$p_x(k) = e^{-\lambda_1}\frac{\lambda_1^k}{k!} \quad \text{for } k = 0, 1, \ldots$$

and

$$p_y(k) = e^{-\lambda_2}\frac{\lambda_2^k}{k!} \quad \text{for } k = 0, 1, \ldots$$

If X and Y are independent, then their sum will have the following distribution: for $n = 0, 1, \ldots$,

$$p_{x+y}(n) = \sum_{k=0}^{\infty} p_x(k)\, p_y(n-k) = \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}\, e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, \lambda_1^k\, \lambda_2^{n-k}$$

Sums of independent random variables.

$$p_{\mathrm{x+y}}(a) \;=\; \sum_x p_{\mathrm{x}}(x)\, p_{\mathrm{y}}(a-x)$$

• **Example (continued).** Let X be Poisson with parameter λ_1 and Y be Poisson with parameter λ_2. If X and Y are independent, then their sum will have the following distribution: for $n = 0, 1, \ldots,$

$$p_{\mathrm{x+y}}(n) \;=\; e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \;=\; \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k}$$

$$=\; \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \;=\; e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}$$

by the Binomial Theorem.

Hence, $X+Y$ is Poisson with parameter $\lambda_1+\lambda_2$.

Sums of independent random variables.

$$f_{x+y}(a) = \int_{-\infty}^{\infty} f_x(x)\, f_y(a-x)\ dx$$

• **Example.** Let X and Y each be uniform over $[0,1]$. Then each will be distributed according to the following probability density function:

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If X and Y are independent, then their sum will have the following continuous distribution:

$$f_{x+y}(a) = \int_{-\infty}^{\infty} f(x)\, f(a-x)\ dx = \int_0^1 f(a-x)dx$$

Observe that $\int_0^1 f(a-x)dx = 0$ whenever $a < 0$ or $a > 2$.

There are two more cases: $0 \le a \le 1$ and $1 \le a \le 2$.

Sums of independent random variables.

• **Example (continued).** $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

If $0 \le a \le 1$,

$$f_{x+y}(a) = \int_0^1 f(a-x)dx = \int_0^a dx = a$$

Now, if $1 \le a \le 2$,

$$f_{x+y}(a) = \int_0^1 f(a-x)dx = \int_{a-1}^1 dx = 2-a$$

Therefore,

$$f_{x+y}(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2-a & \text{if } 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Properties of convolutions.

Convolution of functions f and g is defined

$$f \circ g(a) = \int_{-\infty}^{\infty} f(x)\, g(a-x)\ dx$$

Another notation: $f * g$.

- Convolution is commutative: $f \circ g(a) = g \circ f(a)$
- Convolution is associative: $(f \circ g) \circ h = f \circ (g \circ h)$
- Convolution is distributive: $(f+g) \circ h = f \circ h + g \circ h$
- $f \circ (cg) = c(f \circ g)$ for all $c \in \mathbb{R}$
- If you are familiar with Fourier transforms: $\widehat{f \circ g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

Sums of random variables.

• **Theorem.** Expectation of a sum of random variables is equal to the sum of expectations.

$$E[X+Y] = E[X] + E[Y]$$

Notice: we don't require X and Y to be independent here. The above is true even if they are dependent random variables.

• **Theorem.** If X and Y are independent random variables then,

$$Var(X+Y) = Var(X) + Var(Y)$$

Here we need independence.

• **Example.** Let X be Poisson with parameter λ_1 and Y be Poisson with parameter λ_2. For the case when X and Y are independent, we proved that $X+Y$ is Poisson with parameter $\lambda_1+\lambda_2$. Then

$$E[X] + E[Y] = \lambda_1 + \lambda_2 = E[X+Y]$$

$$Var(X) + Var(Y) = \lambda_1 + \lambda_2 = Var(X+Y)$$

Sums of random variables.

• **Example.** Let X and Y each be uniform over $[0,1]$. For the case when X and Y are independent, we proved that $X+Y$ is distributed according to

$$f_{\mathrm{x+y}}(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2-a & \text{if } 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Check that

$$E[X+Y] = E[X] + E[Y]$$

and

$$Var(X) + Var(Y) = Var(X+Y)$$

Sums of random variables.

• **Example (continued).** Let X and Y each be uniform over $[0,1]$. For the case when X and Y are independent, we proved that $X+Y$ is distributed according to

$$f_{\mathrm{x+y}}(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2-a & \text{if } 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Here $E[X] = E[Y] = \int_0^1 x\,dx = \frac{1}{2}$ while

$$E[X+Y] = \int_{-\infty}^{\infty} x f_{\mathrm{x+y}}(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx$$

$$= \frac{1}{3} + \left[x^2 - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3} = 1 = E[X] + E[Y]$$

Sums of random variables.

• **Example (continued).** Let X and Y each be uniform over $[0, 1]$. For the case when X and Y are independent, we proved that $X + Y$ is distributed according to

$$f_{\mathrm{x+y}}(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2 - a & \text{if } 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Here $Var(X) = E[X^2] - \Big(E[X]\Big)^2 = \int_0^1 x^2 dx - \frac{1}{4} = \frac{1}{12} = Var(Y)$

while $E[X + Y] = 1$, and therefore

$$Var(X+Y) = E[(X+Y)^2]-1 = \int_{-\infty}^{\infty} x^2 f_{\mathrm{x+y}}(x)dx-1 = \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx-1$$

$$= \frac{1}{4} + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 - 1 = \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} - 1 = \frac{1}{6} = Var(X) + Var(Y)$$

Sums of independent random variables.

Consider n **independent identically distributed** (i.i.d.) random variables

$$X_1,\ X_2, \ldots,\ X_n$$

There

$$E[X_1] = E[X_2] = \ldots = E[X_n] = \mu$$

and

$$Var(X_1) = Var(X_2) = \ldots = Var(X_n) = \sigma^2$$

Suppose μ and σ are finite. Then

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n] = n\mu$$

and

$$Var(X_1 + X_2 + \ldots + X_n) = Var(X_1) + Var(X_2) + \ldots + Var(X_n) = n\sigma^2$$

Thus

$$X_1 + X_2 + \ldots + X_n = n\mu \pm \sqrt{n}\sigma$$

and

$$\frac{X_1 + X_2 + \ldots + X_n}{n} = \mu \pm \frac{\sigma}{\sqrt{n}}$$

Sums of independent random variables.

Consider n **independent identically distributed** (i.i.d.) random variables

$$X_1,\ X_2, \ldots,\ X_n$$

There

$E[X_1 + X_2 + \ldots + X_n] = n\mu$ and $Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

Thus

$$X_1 + X_2 + \ldots + X_n = n\mu \pm \sqrt{n}\sigma$$

and

$$\frac{X_1 + X_2 + \ldots + X_n}{n} = \mu \pm \frac{\sigma}{\sqrt{n}}$$

- **Law of Large Numbers.** Given $\epsilon > 0$, then

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) = 0$$

In other words, for n large enough, $\frac{X_1+X_2+\ldots+X_n}{n} \approx \mu$.

Law of Large Numbers.

Consider n **independent identically distributed** (i.i.d.) random variables $X_1,\ X_2, \ldots,\ X_n$. There
$E[X_1 + X_2 + \ldots + X_n] = n\mu$ and $Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

• **Law of Large Numbers.** Given $\epsilon > 0$, then

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) = 0$$

In other words, for n large enough, $\frac{X_1+X_2+\ldots+X_n}{n} \approx \mu$.

Proof: Recall the Chebyshev inequality: for any $\kappa > 0$,

$$P\Big(\Big|X - E[X]\Big| \geq \kappa\Big) \leq \frac{Var(X)}{\kappa^2}$$

Applying it to $X_1 + X_2 + \ldots + X_n$, we obtain

$$P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) = P\Big(|X_1+X_2+\ldots+X_n - n\mu| > n\epsilon\Big)$$

$$\leq \frac{Var(X_1 + X_2 + \ldots + X_n)}{n^2\epsilon^2} = \frac{n\sigma^2}{n^2\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

Central Limit Theorem.

Consider n **independent identically distributed** (i.i.d.) random variables X_1, $X_2, \ldots,$ X_n, each with mean μ and variance σ^2. There

$E[X_1 + X_2 + \ldots + X_n] = n\mu$ and $Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

and therefore $X_1 + X_2 + \ldots + X_n = n\mu \pm \sqrt{n}\sigma$.

Hence, $\frac{X_1+X_2+\ldots+X_n-n\mu}{\sqrt{n}\sigma} = 0 \pm 1$.

In fact, as n gets larger, $\frac{X_1+X_2+\ldots+X_n-n\mu}{\sqrt{n}\sigma}$ is distributed more and more like the standard normal random variable.

- **Central Limit Theorem.** Given $a < b$, then

$$\lim_{n\to\infty} P\left(a \leq \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma} \leq b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

Central Limit Theorem.

- **Central Limit Theorem.** Given $a < b$, then

$$\lim_{n\to\infty} P\left(a \leq \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma} \leq b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx$$

- **Example.** Given $0 \leq p \leq 1$. We say that X is a **Bernoulli random variable** with parameter p if

$$P(X=1) = p \quad \text{and} \quad P(X=0) = 1-p$$

Think of one coin toss. There $E[X] = p$ and $Var(X) = p(1-p)$.

Let $X_1,\ X_2, \ldots,\ X_n$ be independent Bernoulli random variables with the same parameter p. Then

$$S_n = X_1 + X_2 + \ldots + X_n$$

is a Binomial random variable with parameters (n,p). Then

$$P\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \approx \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx$$

Central Limit Theorem.

- **Central Limit Theorem.** Given $a < b$, then

$$\lim_{n\to\infty} P\left(a \le \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

Let $X_1,\ X_2, \ldots,\ X_n$ be independent Bernoulli random variables with the same parameter p. Then

$$S_n = X_1 + X_2 + \ldots + X_n$$

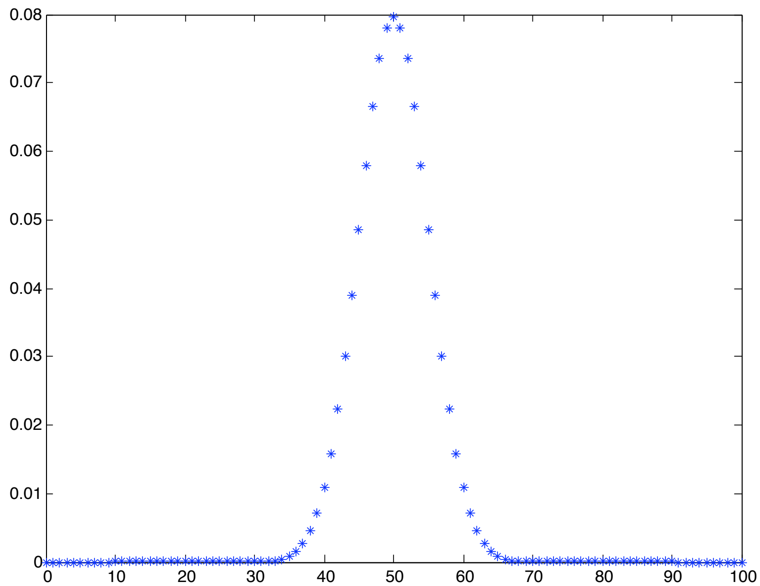
is a Binomial random variable with parameters (n, p). Then, S_n satisfies the following version of the Central Limit Theorem:

- **De Moivre - Laplace Theorem.** Given $a < b$, then

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

Central Limit Theorem.

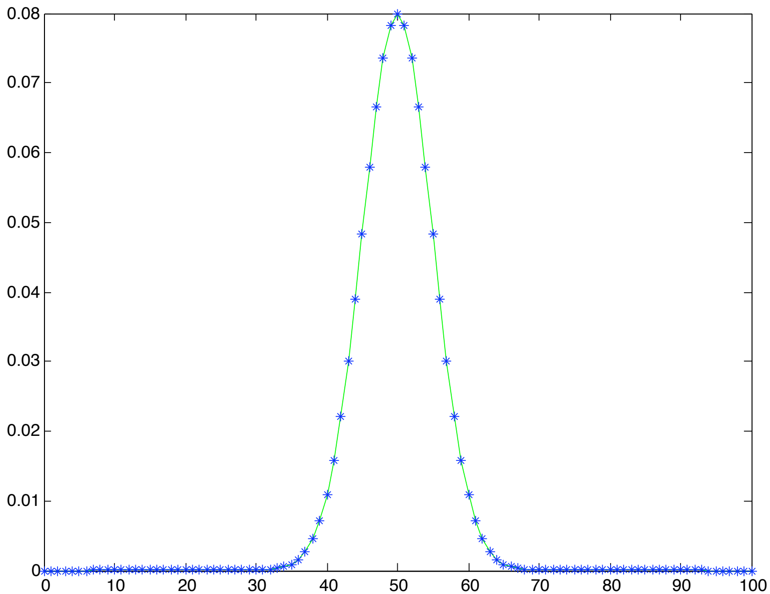
Example. Let S_n be a Binomial random variable with $n = 100$ and $p = \frac{1}{2}$. Estimate $P(S_n = 53)$ and $P(52 \le S_n \le 57)$.



Central Limit Theorem.

Example (continued). Observe that

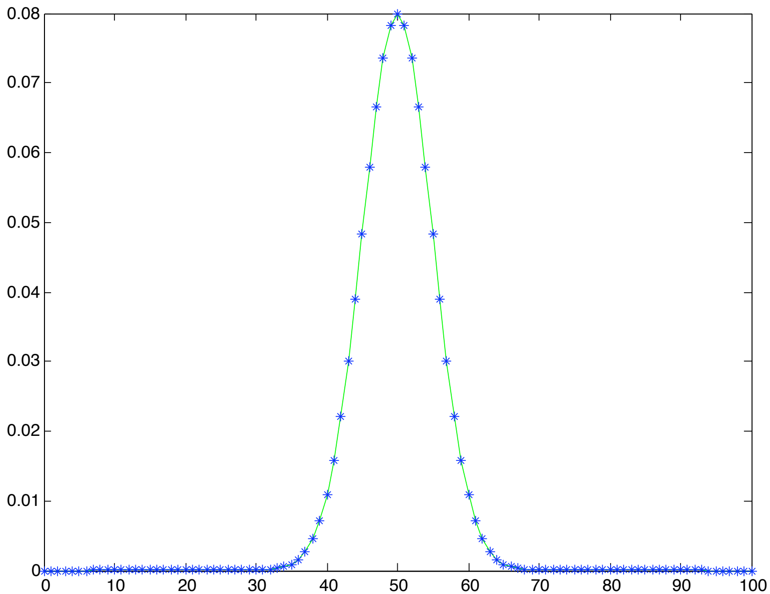
$$P(S_n = 53) = P(52.5 \le S_n \le 53.5)$$



Central Limit Theorem.

Example (continued). $E[S_n] = np = 50$ and

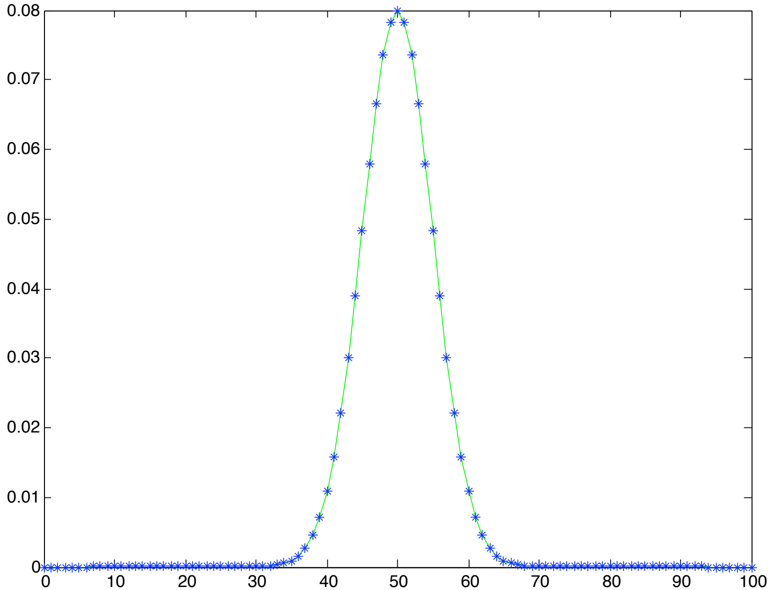
$$P(S_n = 53) = P(52.5 \leq S_n \leq 53.5) = P(2.5 \leq S_n - np \leq 3.5)$$



Central Limit Theorem.

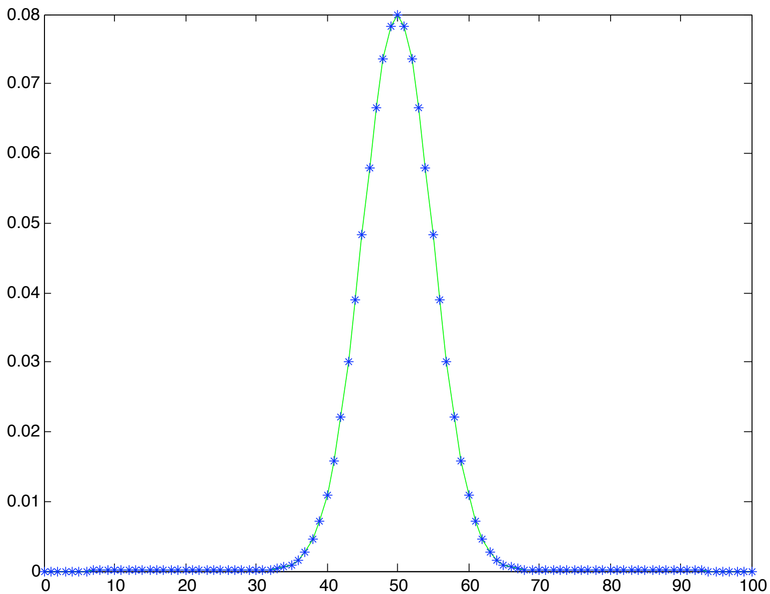
Example (continued). $Var(S_n) = np(1-p) = 25$ and

$$P(S_n = 53) = P(2.5 \le S_n - np \le 3.5) = P\left(\frac{2.5}{5} \le \frac{S_n - np}{\sqrt{np(1-p)}} \le \frac{3.5}{5}\right)$$



Central Limit Theorem.

$$P(S_n = 53) = P\left(0.5 \le \frac{S_n - np}{\sqrt{np(1-p)}} \le 0.7\right) \approx \int_{0.5}^{0.7} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$



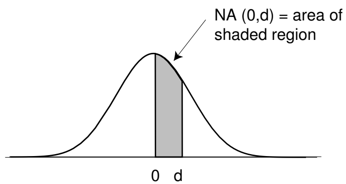
Central Limit Theorem.

$$P(S_n = 53) = P\left(0.5 \le \frac{S_n - np}{\sqrt{np(1-p)}} \le 0.7\right) \approx \int_{0.5}^{0.7} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{0}^{0.7} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx - \int_{0}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 0.2580\ldots - 0.1915\ldots = 0.0665\ldots$$

Appendix A

Normal distribution table

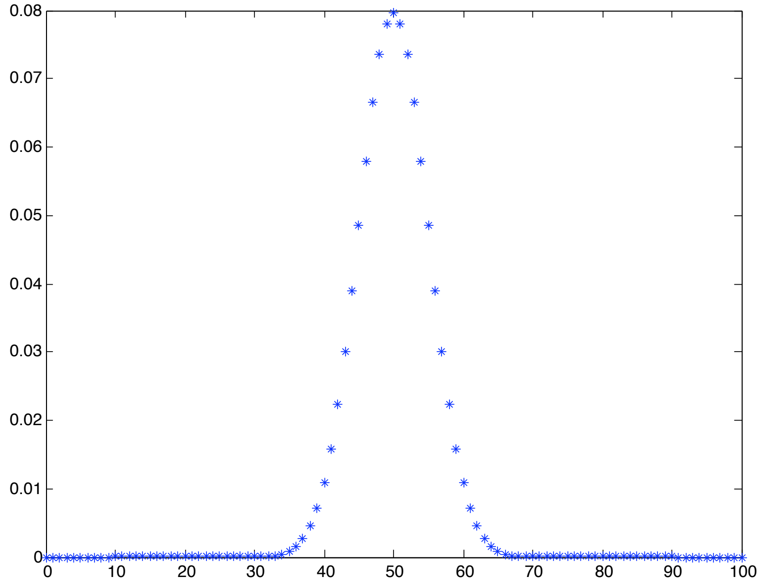


	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830

Central Limit Theorem.

Example (continued). So $P(S_n = 53) \approx$ 0.0665...

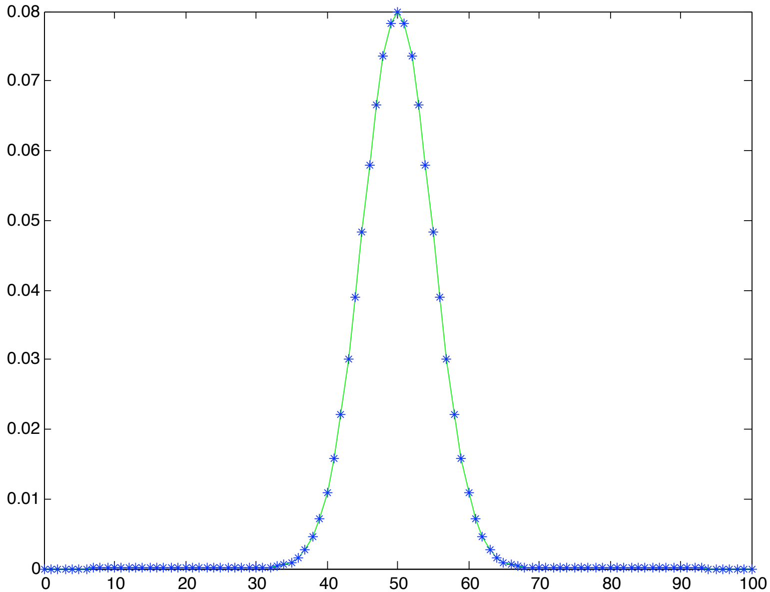
Next, estimate $P(52 \leq S_n \leq 57)$.



Central Limit Theorem.

Example (continued). Observe that

$$P(52 \le S_n \le 57) = P(51.5 \le S_n \le 57.5)$$



Central Limit Theorem.

Example (continued). Recall that $E[S_n] = np = 50$ and $Var(S_n) = np(1-p) = 25$. Thus

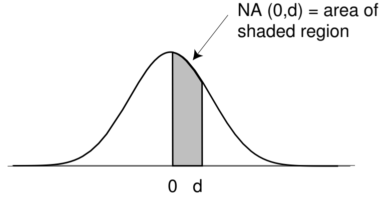
$$P(52 \leq S_n \leq 57) = P(51.5 \leq S_n \leq 57.5) = P(1.5 \leq S_n - np \leq 7.5)$$

$$= P\left(\frac{1.5}{5} \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq \frac{7.5}{5}\right) = P\left(0.3 \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq 1.5\right)$$

$$\approx \int_{0.3}^{1.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \int_{0}^{1.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx - \int_{0}^{0.3} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= 0.4332\ldots - 0.1179\ldots = 0.3153\ldots$$

Normal distribution table



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

de Moivre's and Stirling's formulas. It is easy to check that $n! < n^n$. A better asymptotic analysis of $n!$ comes in the form of Stirling's formula, and its earlier version de Moivre's formula.

de Moivre's Formula. *There exists* $L \in (0, \infty)$ *such that*

$$n! \sim \; L \; n^{n+\frac{1}{2}} e^{-n}$$

in the following sense

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} = L$$

Proof: Let $x_n = \dfrac{n!}{n^{n+\frac{1}{2}} e^{-n}}$. Then $x_1 = e$,

$$\ln \frac{x_j}{x_{j-1}} = 1 + \left(j - \frac{1}{2}\right) \ln\left(1 - \frac{1}{j}\right) = 1 - \left(j - \frac{1}{2}\right)\left(\frac{1}{j} + \frac{1}{2j^2} + \frac{1}{3j^3} + \dots\right) \sim -\frac{1}{12j^2},$$

and $\displaystyle x_n = \exp\left\{1 + \sum_{j=2}^{n} \ln \frac{x_j}{x_{j-1}}\right\}$, where $\displaystyle \sum_{j=2}^{n} \ln \frac{x_j}{x_{j-1}} \to \sum_{j=2}^{\infty} \ln \frac{x_j}{x_{j-1}} \sim \sum_{j=2}^{\infty} \frac{1}{12j^2} < \infty$

Stirling's Formula. $L = \sqrt{2\pi}$, i.e. $n! \sim \; \sqrt{2\pi} \; n^{n+\frac{1}{2}} e^{-n}$

Bernoulli LLN.

Consider n i.i.d. random variables $X_1,\ X_2, \ldots,\ X_n$.

- **Law of Large Numbers.** Given $\epsilon > 0$, then

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) = 0$$

Let $X_1,\ X_2, \ldots,\ X_n$ be independent Bernoulli random variables with the same parameter p. Then

$$S_n = X_1 + X_2 + \ldots + X_n$$

is a Binomial random variable with parameters (n, p).

- **Bernoulli's Law of Large Numbers.** *If S_n be a Binomial random variable with fixed parameters n and p, then for any $\epsilon > 0$,*

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While LLN can be proved via Chebyshev inequality, in the 18th century, Bernoulli LLN was proved by using the following equations:

- $\sum_{k=1}^{n} k\binom{n}{k}p^k q^{n-k} = np$
- $\sum_{k=2}^{n} k(k-1)\binom{n}{k}p^k q^{n-k} = n(n-1)p^2$
- $\sum_{k=0}^{n} (k-np)^2\binom{n}{k}p^k q^{n-k} = npq$

where $q = 1 - p$ and the top two formulas are proved by differentiating $(px+q)^n = \sum_{k=0}^{n} \binom{n}{k}x^k p^k q^{n-k}$.

Historical Remark (Grinstead & Snell, p.310). Bernoulli's Law of Large Numbers was first proved by the Swiss mathematician James Bernoulli in the fourth part of his work *Ars Conjectandi* published posthumously in 1713. As often happens with a first proof, BernoulliÕs proof was much more difficult than the proof we have presented using ChebyshevÕs inequality.

Chebyshev developed his inequality to prove a general form of the Law of Large Numbers. The inequality itself appeared much earlier in a work by Bienaymé, and in discussing its history Maistrov remarks that it was referred to as the Bienaymé-Chebyshev Inequality for a long time.

de Moivre – Laplace CLT.

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- **De Moivre - Laplace Theorem.** Given $a < b$, then

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

So, for n large enough,

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \approx \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

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So, for n large enough,

$$P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right)\ \approx\ \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\ dx$$

de Moivre – Laplace CLT.

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Proof: Let $q = 1 - p$. Then

$$P(np + a\sqrt{npq} < S_n \le np + b\sqrt{npq}) = \sum_{np+a\sqrt{npq}<k\le np+b\sqrt{npq}} \binom{n}{k} p^k q^{n-k}$$

Let $j = k - np$. Then $np + a\sqrt{npq} < k \le np + b\sqrt{npq}$

implies $a\sqrt{npq} < j \le b\sqrt{npq}$, and by Stirling's Formula,

$$\binom{n}{k} \sim \frac{1}{L\sqrt{npq}} \left(p + \frac{j}{n}\right)^{-j-np} \left(q - \frac{j}{n}\right)^{j-nq} = \frac{1}{L\sqrt{npq}} \left(1 + \frac{j}{np}\right)^{-j-np-1/2} \left(1 - \frac{j}{nq}\right)^{j-nq-1/2} p^{-k} q^{k-n}$$

and

$$\binom{\mathbf{n}}{\mathbf{k}} \mathbf{p}^{\mathbf{k}} \mathbf{q}^{\mathbf{n-k}} \sim \frac{1}{\sqrt{2\pi npq}} \exp\left\{-\frac{\mathbf{j}^2}{\mathbf{2npq}}\right\}$$

de Moivre – Laplace CLT.

$$\lim_{n\to\infty} P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

Proof (continued): $P\left(a < \frac{S_n-np}{\sqrt{npq}} \le b\right) = \sum_{np+a\sqrt{npq}<k\le np+b\sqrt{npq}} \binom{n}{k} p^k q^{n-k}$

Let $j = k - np$. Then $a\sqrt{npq} < j \le b\sqrt{npq}$, and $\binom{\mathbf{n}}{\mathbf{k}}\mathbf{p^k q^{n-k}} \sim \frac{\mathbf{1}}{\sqrt{2\pi npq}} \exp\left\{-\frac{\mathbf{j^2}}{\mathbf{2npq}}\right\}$

Thus $P\left(a < \frac{S_n-np}{\sqrt{np(1-p)}} \le b\right) \sim \sum_{a\sqrt{npq}<j\le b\sqrt{npq}} \frac{1}{\sqrt{2\pi npq}} \exp\left\{-\frac{j^2}{2npq}\right\}$

$$= \sum_{a\sqrt{npq}<j\le b\sqrt{npq}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(j\Delta)^2}{2}\right\}\cdot\Delta \longrightarrow \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

as $n \to \infty$, where $\Delta = \frac{1}{\sqrt{npq}}$.

de Moivre – Laplace CLT.

$$\lim_{n\to\infty} P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

Proof (continued):

$$P\left(a < \frac{S_n-np}{\sqrt{np(1-p)}} \le b\right) \sim \sum_{a\sqrt{npq}<j\le b\sqrt{npq}} \frac{1}{\sqrt{2\pi npq}} \exp\left\{-\frac{j^2}{2npq}\right\}$$

$$= \sum_{a\sqrt{npq}<j\le b\sqrt{npq}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(j\Delta)^2}{2}\right\}\cdot\Delta \longrightarrow \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx$$

as $n \to \infty$, where $\Delta = \frac{1}{\sqrt{npq}}$.

Hence

$$\frac{P\left(a < \frac{S_n-np}{\sqrt{np(1-p)}} \le b\right)}{\int_a^b \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx} \longrightarrow 1 \quad \text{as } n \to \infty.$$