MTH 463/563

Lectures 18 - 22

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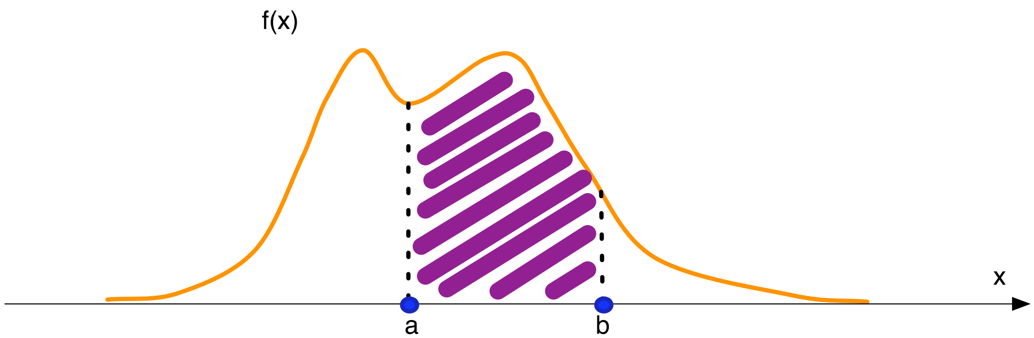
Topics:

- Continuous random variables.
- Exponential random variables.
- Uniform random variable.
- Normal random variable.
- Expectation of continuous variables.
- Variance of continuous variables.
- Functions of a random variable.

Continuous random variables.

Definition. We say that X is a **continuous random variable** if there exists a nonnegative function $f(x)$ defined for all real x such that for any $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)dx$$



Such function $f(x)$ is the probability density function of X.

Continuous random variables.

Definition. We say that X is a **continuous random variable** if there exists a nonnegative function $f(x)$ defined for all real x such that for any $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Properties:

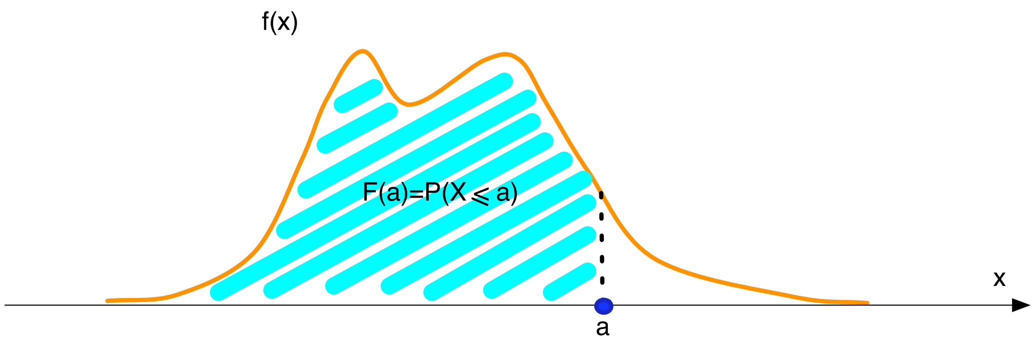
- $\int\limits_{-\infty}^{\infty} f(x)dx = P(-\infty < X < \infty) = 1$, where $\int\limits_{-\infty}^{\infty} f(x)dx = \lim\limits_{a\to\infty} \int\limits_{-a}^{a} f(x)dx$
- $P(X = a) = \int\limits_{a}^{a} f(x)dx = 0$ for any real a.
- Hence

$$P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b) = \int_a^b f(x)dx$$

Continuous random variables.

Definition. Let X be a continuous random variable with density function $f(x)$. Then its cumulative distribution function (cdf) is

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$



- Here, for any $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx = F(b) - F(a)$$

Continuous random variables.

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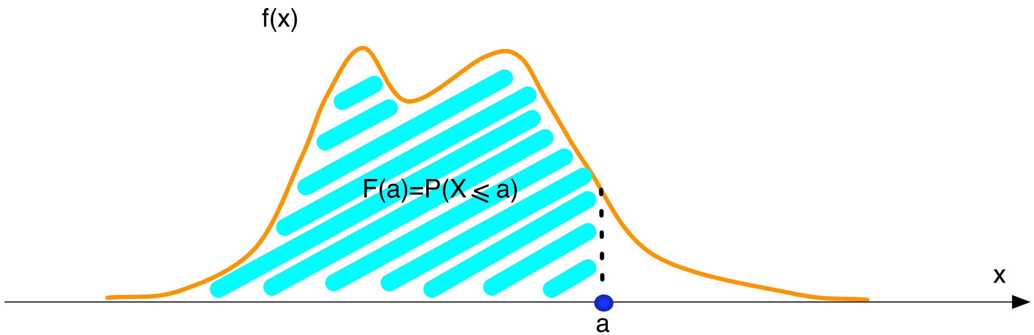


- Here, for any real a, the derivative

$$F'(a) = \frac{d}{da}\int_{-\infty}^{a} f(x)dx = f(a)$$

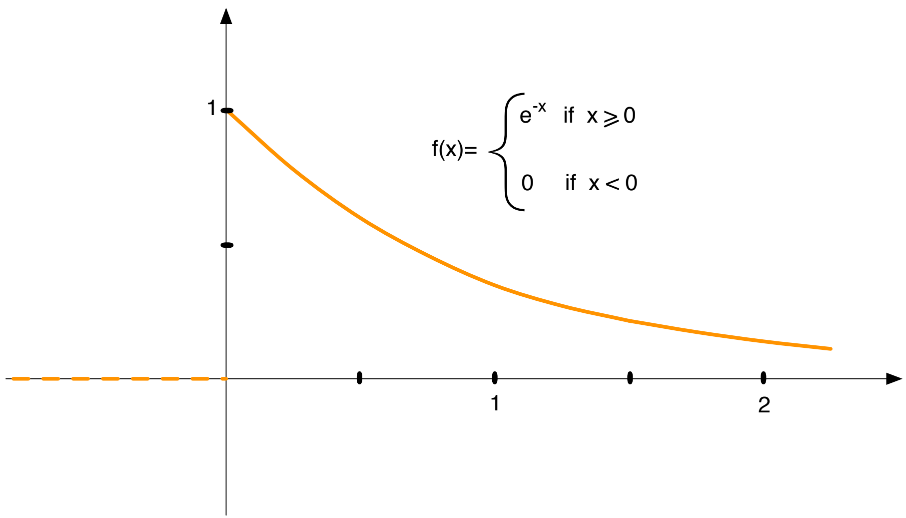
vis the Fundamental Theorem of Calculus.

Continuous random variables.

• **Example.** Let X be a continuous random variable with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.



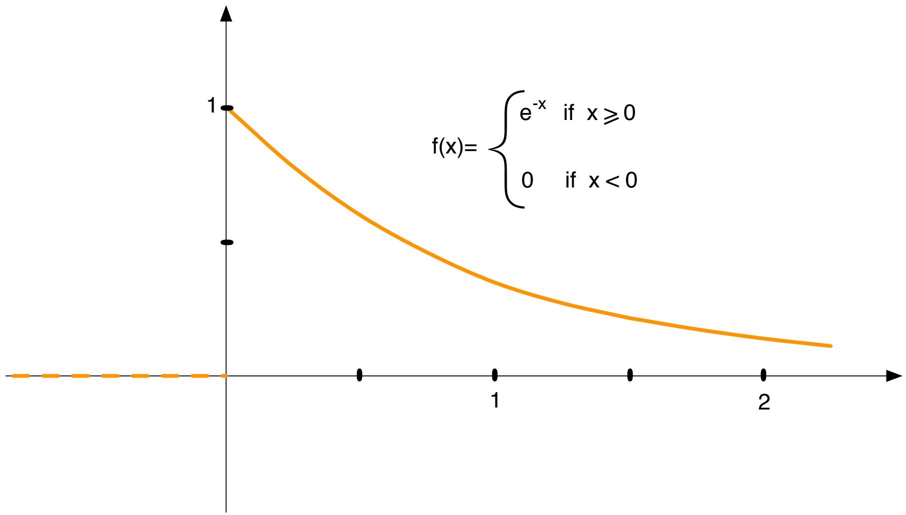
Continuous random variables.

• **Example (continued).** We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.

Solution: Here $f(x)$ is nonnegative and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} e^{-x}dx = 0 + \Big[- e^{-x} \Big]_{0}^{\infty} = 1$$

Thus $f(x)$ is indeed a probability density function.

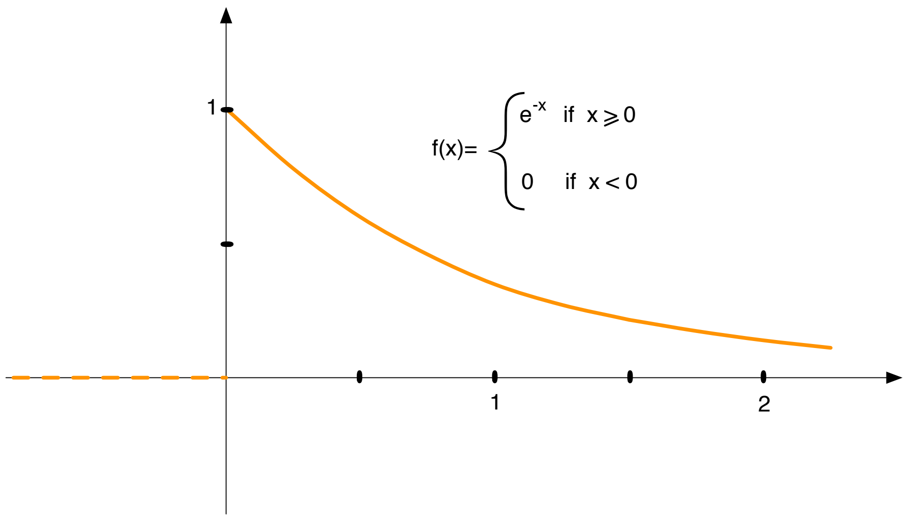


Continuous random variables.

• **Example (continued).** We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.

Solution:

$$P(1 < X < 2) = \int_1^2 f(x)dx = \int_1^2 e^{-x}dx = \left[-e^{-x}\right]_1^2 = e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = 0.232544158\ldots$$

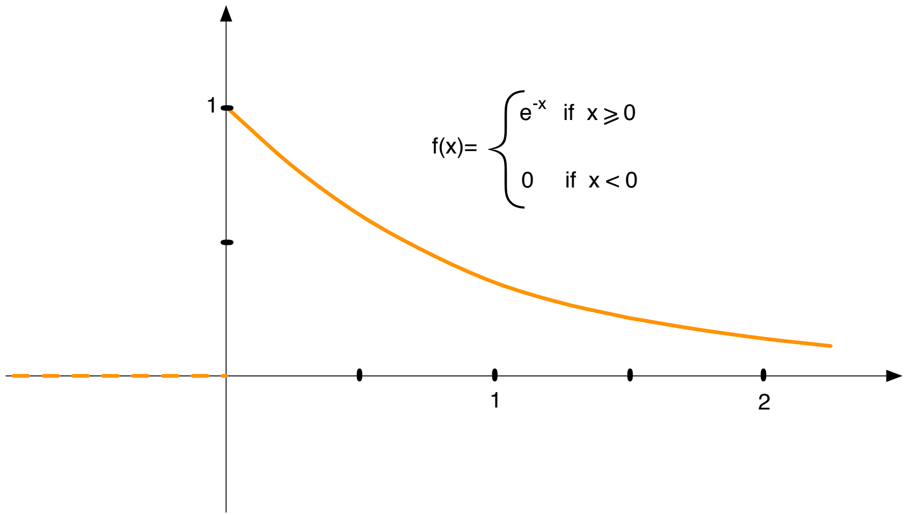


Exponential random variables.

• Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an exponential random variable with parameter λ.



Exponential random variables.

• Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an exponential random variable with parameter λ.

• Check:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} \lambda e^{-\lambda x} dx = 0 + \Big[- e^{-\lambda x} \Big]_0^{\infty} = 1$$

Thus $f(x)$ is indeed a probability density function.

• Note: Exponential random variable is a continuous analogue to the geometric random variable. In particular it also satisfies the following *memorylessensss* property:

$$P(X > a + b \mid X > a) = P(X > b)$$

for any two positive a and b.

Uniform random variables.

• Consider an interval $[\alpha, \beta]$, where $\alpha < \beta$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Then X is said to be an uniform random variable over $[\alpha, \beta]$.

• Check:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\alpha} 0 \cdot dx + \int_{\alpha}^{\beta} \frac{dx}{\beta-\alpha} + \int_{\beta}^{\infty} 0 \cdot dx = 0 + \left[\frac{x}{\beta-\alpha}\right]_{\alpha}^{\beta} + 0$$

$$= \frac{\beta}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} = 1$$

Thus $f(x)$ is indeed a probability density function.

Continuous random variables.

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **expectation** is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\ dx$$

Properties:

• For any real valued function g, $g(X)$ will also be a **random variable**, and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\ dx$$

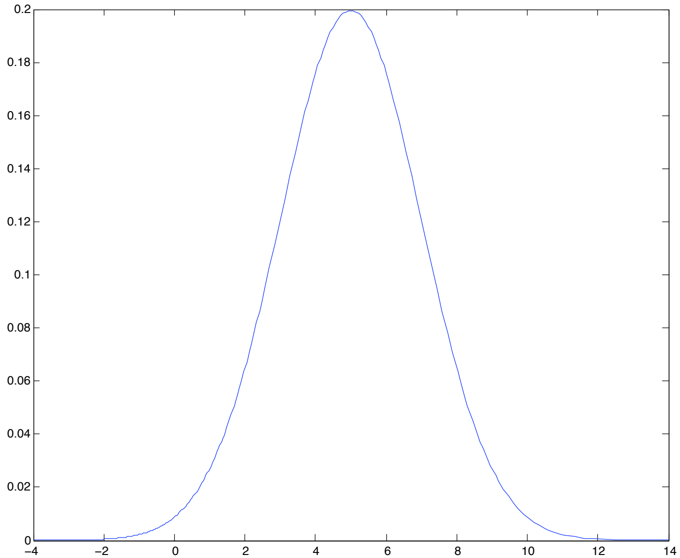
• **Markov inequality.** If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

$$P(X \geq \alpha) \ \leq \ \frac{E[X]}{\alpha}$$

• **Chebyshev inequality.** If X is a random variable with finite mean μ and variance, then for any $\kappa > 0$,

$$P(|X - \mu| \geq \kappa) \ \leq \ \frac{Var(X)}{\kappa^2}$$

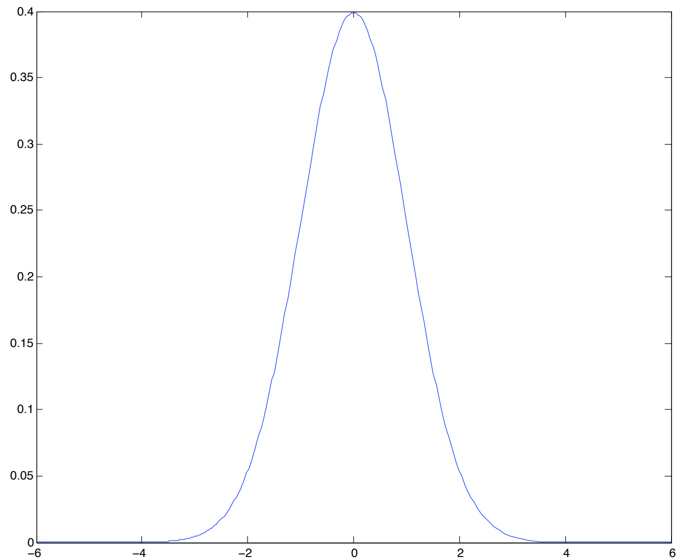
Normal random variables.



X is a **Normal (Gaussian) random variable** with parameters μ and σ^2 if its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

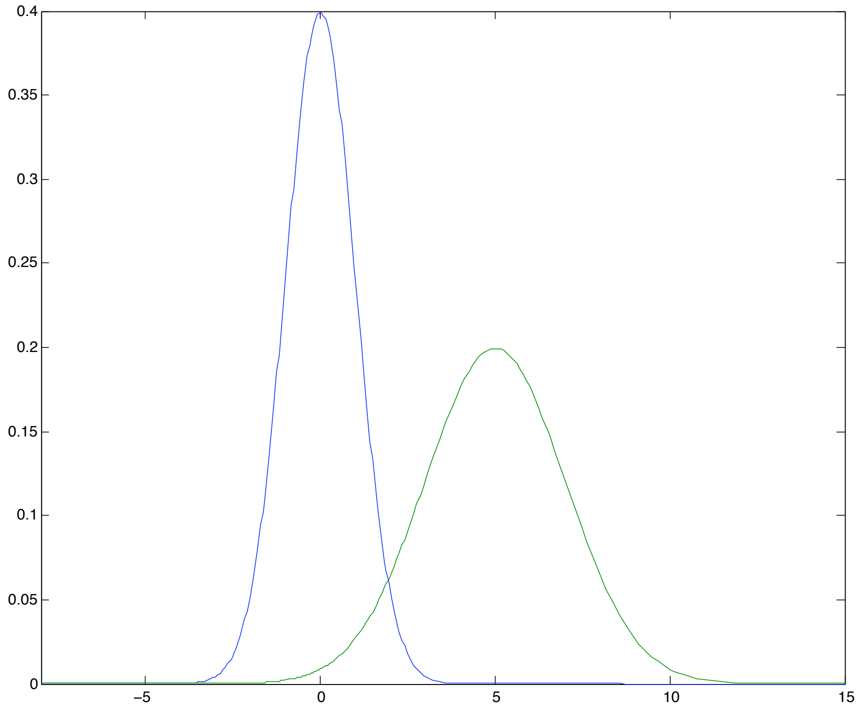
Normal random variables.



A normal random variable X is said to be **standard normal** if its parameters $\mu = 0$ and $\sigma^2 = 1$, and therefore

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

Normal random variables.



$\mathcal{N}(\mu, \sigma^2)$ denotes normal distribution with parameters μ and σ^2. Here I plotted two normal densities, one standard normal $\mathcal{N}(0, 1)$ and one $\mathcal{N}(5, 4)$.

Normal random variables.

We need to check that the standard normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

integrates to 1.

Here let $I = \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. Then

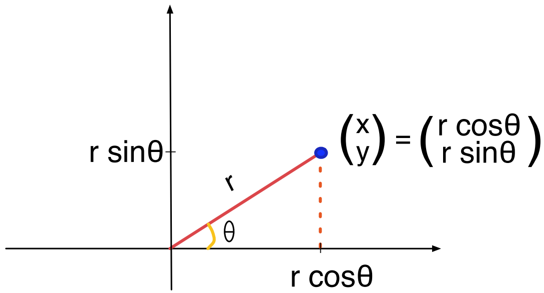
$$I^2 = \int\limits_{-\infty}^{\infty} \left(e^{-\frac{y^2}{2}} \cdot I\right) dy = \int\limits_{-\infty}^{\infty} \left(e^{-\frac{y^2}{2}} \cdot \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right) dy = \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot e^{-\frac{x^2}{2}} dx\right) dy$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot e^{-\frac{x^2}{2}}\, dx\, dy = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy$$

Normal random variables.

We let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$, and showed that $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} \, dx \, dy$

Use **polar coordinates**: let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \le \theta < 2\pi$ and $0 \le r < \infty$.



Here

$$dx\,dy = |J| dr\,d\theta = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| dr\,d\theta = \left| \det \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \right| dr\,d\theta = r\,dr\,d\theta$$

and

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} \, dx\,dy = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} \, r\,dr\,d\theta = \int_0^{2\pi} \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty} d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Normal random variables.

So $I^2 = 2\pi$ and $I = \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

Hence, $\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

Check: For a general $\mathcal{N}(\mu, \sigma^2)$ random variable,

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1,$$

where we let $y = \frac{x-\mu}{\sigma}$.

Expectation and variance.

Recall

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **expectation** is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\ dx$$

Then

- For any real valued function g, $g(X)$ will also be a **random variable**, and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\ dx$$

and

- Given constants α and β,

$$E[\alpha X + \beta] = \alpha E[X] + \beta$$

Expectation and variance.

Now, let X be a random variable with mean $E[X] = \mu$.

- **Definition.** The **variance** of X is

$$Var(X) = E\Big[(X-\mu)^2\Big]$$

- **Definition.** The **standard deviation** of X is

$$SD(X) = \sqrt{Var(X)} = \sqrt{E\Big[(X-\mu)^2\Big]}$$

Finally,

- **Theorem.** The **variance** of X is

$$Var(X) = E[X^2] - \mu^2$$

Exponential random variables.

Example. Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an exponential random variable with parameter λ.

We want to find its expectation.

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{0}^{\infty} \lambda x \cdot e^{-\lambda x} dx$$

- Recall the **integration by parts formula,**

$$\int uv' dx = uv - \int u'v dx$$

Here we let $u(x) = x$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 1$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int\limits_0^\infty \lambda x \cdot e^{-\lambda x} dx = \int\limits_0^\infty u(x)v'(x) dx$

Exponential random variables.

Example (continued).

- Recall the **integration by parts formula,**

$$\int uv'dx = uv - \int u'vdx$$

Here we let $u(x) = x$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 1$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int_0^\infty \lambda x \cdot e^{-\lambda x}dx = \int_0^\infty u(x)v'(x)dx$

$$E[X] = \int_0^\infty \lambda x \cdot e^{-\lambda x}dx = \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty (-e^{-\lambda x})dx = \int_0^\infty e^{-\lambda x}dx$$

$$= \frac{1}{\lambda} \cdot \int_0^\infty \lambda e^{-\lambda x}dx = \frac{1}{\lambda}$$

as we have shown that $\int_{-\infty}^\infty f(x)dx = \int_0^\infty \lambda e^{-\lambda x}dx = 1.$

Normal random variables.

Example. Let X be a **Normal (Gaussian) random variable** with parameters μ and σ^2. Its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

Then

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \; + \mu \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \; + \mu$$

Normal random variables.

Example (continued). Let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx + \mu = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \cdot e^{-\frac{y^2}{2}} \, dy + \mu = \mu$$

since $y \cdot e^{-\frac{y^2}{2}}$ is an **odd** function, implying

$$\int_{-\infty}^{0} y \cdot e^{-\frac{y^2}{2}} \, dy = -\int_{0}^{\infty} y \cdot e^{-\frac{y^2}{2}} \, dy$$

Recall that $g(x)$ is an odd function if $g(-x) = -g(x)$.

Exponential random variables.

Example. Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an exponential random variable with parameter λ. We know its expectation $E[X] = \frac{1}{\lambda}$.

We want to find its variance, $Var(X) = E[X^2] - \Big(E[X]\Big)^2$

- Here

$$Var(X) = E[X^2] - \frac{1}{\lambda^2} = \int_0^\infty \lambda x^2 \cdot e^{-\lambda x} dx - \frac{1}{\lambda^2}$$

Once again, recall the **integration by parts formula,**

$$\int uv'\ dx = uv - \int u'v\ dx$$

Here we let $u(x) = x^2$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 2x$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int\limits_0^\infty \lambda x^2 \cdot e^{-\lambda x} dx = \int\limits_0^\infty u(x)v'(x) dx$

Exponential random variables.

Example (continued). Once again, recall the **integration by parts formula,**

$$\int uv'\ dx = uv - \int u'v\ dx$$

Here we let $u(x) = x^2$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 2x$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int\limits_0^\infty \lambda x^2 \cdot e^{-\lambda x} dx = \int\limits_0^\infty u(x)v'(x)dx$

$$Var(X) = \int_0^\infty \lambda x^2 \cdot e^{-\lambda x} dx - \frac{1}{\lambda^2} = \left[-x^2 e^{-\lambda x}\right]_0^\infty - \int_0^\infty (-2xe^{-\lambda x}) dx - \frac{1}{\lambda^2}$$

$$= 2\int_0^\infty x e^{-\lambda x} dx - \frac{1}{\lambda^2} = \frac{2}{\lambda}\int_0^\infty x \cdot \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2} = \frac{2}{\lambda} \cdot E[X] - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Normal random variables.

Example. Let X be a **Normal (Gaussian) random variable** with parameters μ and σ^2. Its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

We have proved that $E[X] = \mu$.

We want to find its variance, $Var(X)$

• We let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$Var(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}\, dy$$

Normal random variables.

Example (continued). We let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$Var(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}\, dy$$

Now, let $u(y) = y$ and $v(y) = -e^{-\frac{y^2}{2}}$. Then $u'(y) = 1$ and $v'(y) = y \cdot e^{-\frac{y^2}{2}}$, and $\int\limits_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}\, dy = \int\limits_{-\infty}^{\infty} u(y)v'(y)dy$

Integrating by parts, we obtain

$$Var(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}\, dy = \frac{\sigma^2}{\sqrt{2\pi}} \left[-ye^{-\frac{y^2}{2}} \right]_{-\infty}^{\infty} - \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-e^{-\frac{y^2}{2}} \right) dy$$

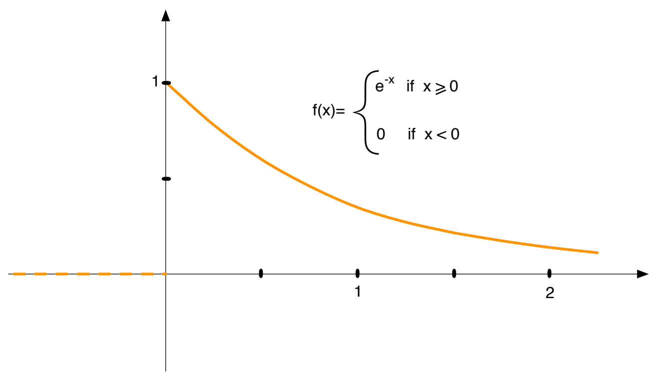
$$= \sigma^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy = \sigma^2$$

Expectation and variance.

• Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an exponential random variable with parameter λ.

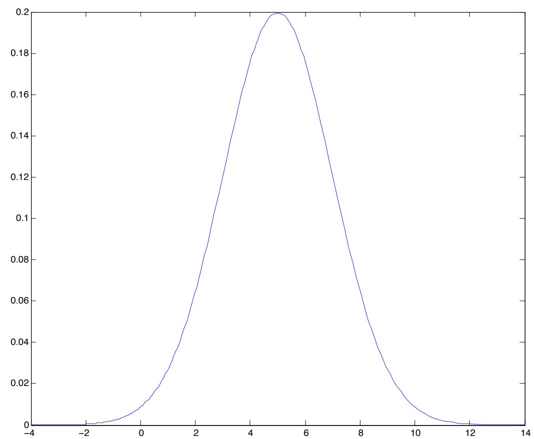


Here $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$

Expectation and variance.

X is a **Normal (Gaussian) random variable** with parameters μ and σ^2 if its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$



Here $E[X] = \mu$ and $Var(X) = \sigma^2$

Distribution of a function of a random variable.

Example. Let X be a continuous random variable uniformly distributed over $(0, 1)$. Take $\lambda > 0$, and let $Y = -\frac{1}{\lambda} \ln X$.

Then $0 < Y < \infty$, and the cumulative distribution function of Y is

$$F_Y(y) = P(Y \leq y) = P\left(-\frac{1}{\lambda} \ln X \leq y\right) = P(\ln X \geq -\lambda y)$$

$$= P(X \geq e^{-\lambda y}) = 1 - e^{-\lambda y}$$

if $y > 0$.

Differentiating $F_Y(y)$ we obtain the density function of Y

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence $Y = -\frac{1}{\lambda} \ln X$ is an exponential random variable with parameter λ.

Distribution of a function of a random variable.

Example. Let X be a continuous random variable uniformly distributed over $(0, 1)$. Let $Y = X^n$.

Then $0 \leq Y \leq 1$, and the cumulative distribution function of Y is

$$F_Y(y) = P(Y \leq y) = P(X^n \leq y) = P(X \leq y^{1/n}) = y^{1/n}$$

if $0 \leq y \leq 1$.

Differentiating $F_Y(y)$ we obtain the density function of Y

$$f_Y(y) = \begin{cases} \frac{1}{n} \cdot y^{1/n-1} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Distribution of a function of a random variable.

Example. Let X be a continuous random variable with density function $f_X(x)$. Let $Y = X^2$.

Then for $y \geq 0$, the cumulative distribution function of Y is

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Differentiating $F_Y(y)$ we obtain the density function of Y

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\Big[f_X(\sqrt{y}) + f_X(-\sqrt{y})\Big] & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Distribution of a function of a random variable.

Theorem. Let X be a continuous random variable with density function $f_X(x)$. If $g(x)$ is a strictly monotone (increasing or decreasing) differentiable function, and if $Y = g(X)$, then the probability density function of Y

$$f_Y(y) = \begin{cases} f_X\Big(g^{-1}(y)\Big) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \text{ s.t. } f_X(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where g^{-1} is the inverse of g: $g(x) = y \Leftrightarrow g^{-1}(y) = x$.

- **Proof:** W.l.o.g. let $g(x)$ be an increasing function.

For y in the range of g (i.e. $y = g(x)$ for some x s.t. $f_X(x) \neq 0$),

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\Big(g^{-1}(g(X)) \leq g^{-1}(y)\Big)$$

$$= P(X \leq g^{-1}(y)) = F_X\Big(g^{-1}(y)\Big),$$

and differentiate, obtaining, $f_Y(y) = f_X\Big(g^{-1}(y)\Big) \cdot \frac{d}{dy} g^{-1}(y)$.

If y is not in the range of g, then $F_Y(y) = 0$ or 1, and $f_Y(y) = 0$.