MTH 463/563 - Lecture 17

Review problems.

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Review problems.

• **Problem.** If 17 people are to be divided into two committees of respective sizes 5, and 12, how many divisions are possible? Here each person can serve only on one committee.

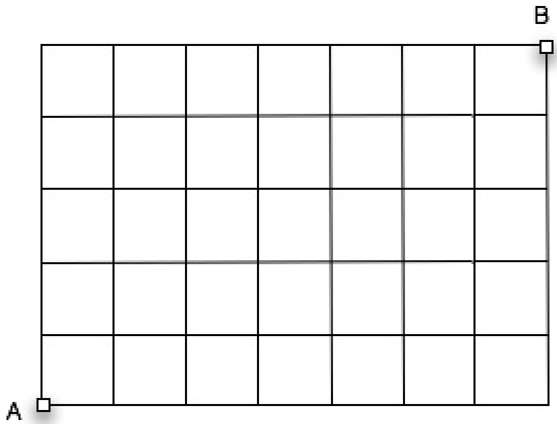
Solution: We are splitting 17 people into two groups: committee A of five and committee B of 12. Since the 17 people are different individuals, it is the same as making 17-long strings of 5 A's and 12 B's. Thus there are

$$\binom{17}{5}$$

ways to do so.

Review problems.

• **Problem.** Consider a walk on the grid pictured below, originating at the point labelled **A**. Each time the walker can go one step up or one step to the right . This is continued until the point labeled **B** is reached. How many different paths from **A** to **B** are possible? Here is an example of such path: `Up-Up-Right-Right-Right-Up-Right-Right-Up-Right-Up-Right`



Solution: Each distinct path corresponds to a distinct string made of 5 U's and 7 R's, where U and R stand for `Up` and `Right` respectively. Thus there are $\binom{12}{5}$ such paths.

Review problems.

- **Problem.** Use the Binomial Theorem to show

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

Solution: Recall the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging in $x = -1$ and $y = 1$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n = (-1+1)^n = 0$$

Review problems.

- **Problem.** Compute $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$

Solution: Recall the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging in $x = 1$ and $y = 2$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n = (1+2)^n = 3^n$$

Review problems.

- **Problem.** $k \cdot \binom{n}{k} - n \cdot \binom{n-1}{k-1} = \ ?$

Solution:

$$k \cdot \binom{n}{k} - n \cdot \binom{n-1}{k-1} = \frac{n!}{(n-k)!(k-1)!} - \frac{n!}{(n-k)!(k-1)!} = 0$$

- **Problem.** Solve $\sum_{k=0}^{7} \binom{7}{k} x^{2k} = 2^7 x^7$ for x.

Solution:

Binomial theorem $\Rrightarrow \sum_{k=0}^{7} \binom{7}{k} x^{2k} = (x^2+1)^7$.

Thus $(x^2+1)^7 = (2x)^7$. Thus $x = 1$.

Review problems.

• **Problem.** In order to compute $\sum_{k=0}^{n} \binom{2n+1}{2k}$ one has to perform the following three steps:

(i) Find $\sum_{i=0}^{2n+1} \binom{2n+1}{i}$

Solution:

Binomial theorem $\Rrightarrow \sum_{i=0}^{2n+1} \binom{2n+1}{i} = (1+1)^{2n+1} = 2^{2n+1}$

(ii) Find $\sum_{i=0}^{2n+1} (-1)^i \binom{2n+1}{i}$

Solution:

Binomial theorem $\Rrightarrow \sum_{i=0}^{2n+1} (-1)^i \binom{2n+1}{i} = ((-1)+1)^{2n+1} = 0$

Review problems.

(iii) Notice that $\frac{1+(-1)^i}{2}$ is equal to one when i is even and zero when i is odd. Use steps (i) and (ii) to complete the following computation:

$$\sum_{k=0}^{n}\binom{2n+1}{2k} = \sum_{i=0,2,4,\ldots,2n}\binom{2n+1}{i} = \sum_{i=0}^{2n+1}\frac{1+(-1)^i}{2}\binom{2n+1}{i}$$

Solution:

$$\sum_{k=0}^{n}\binom{2n+1}{2k} = \sum_{i=0}^{2n+1}\frac{1+(-1)^i}{2}\binom{2n+1}{i}$$

$$= \frac{1}{2}\sum_{i=0}^{2n+1}\binom{2n+1}{i} + \frac{1}{2}\sum_{i=0}^{2n+1}(-1)^i\binom{2n+1}{i}$$

$$= \frac{1}{2}\cdot 2^{2n+1} = 2^{2n} = 4^n$$

Review problems.

- **Problem.** In order to compute $\sum_{k=0}^{n}\binom{2n+1}{2k}$ one has to perform the following three steps:

(i) Find $\sum_{i=0}^{2n+1}\binom{2n+1}{i}$

(ii) Find $\sum_{i=0}^{2n+1}(-1)^i\binom{2n+1}{i}$

(iii) Notice that $\frac{1+(-1)^i}{2}$ is equal to one when i is even and zero when i is odd. Use steps (i) and (ii) to complete the following computation:

$$\sum_{k=0}^{n}\binom{2n+1}{2k} = \sum_{i=0,2,4,\dots,2n}\binom{2n+1}{i} = \sum_{i=0}^{2n+1}\frac{1+(-1)^i}{2}\binom{2n+1}{i}$$

Review problems.

• **Problem.** If a fair die is rolled five times, what is the probability that 6 comes up exactly three times?

Solution: Since this is a fair die, the probability that 6 comes up when the die is rolled is

$$p = \frac{1}{6}$$

So, we perform $n = 5$ independent experiments with probability $p = \frac{1}{6}$ of success. Then the probability of exactly $k = 3$ successes in the $n = 5$ trials is Binomial

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} = \binom{5}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2$$

$$= \binom{5}{3} \cdot \frac{25}{6^5} = \frac{125}{3,888} = 0.03\overline{2150}$$

Alternatively, it can be solved via counting.

Review problems.

• **Example: Birthday Problem.** Find the probability that among n persons, at least two have birthdays on the same day (but not necessarily in the same year). Assume all days of the year are equally likely to be one's birthday, and ignore February 29th.

Solution: There are two cases: when $n \leq 365$ and when $n > 365$.

If $n \leq 365$, then we find the probability that they all have different birthdays: $\frac{365 \cdot 364 \ldots (365-n+1)}{365^n}$

and subtract it from 1, obtaining

$$1 - \frac{365 \cdot 364 \ldots (365 - n + 1)}{365^n}$$

When $n > 365$, this probability is equal to 1.

Review problems.

• **Problem.** Independent trials that result in a success with probability p and failure with probability $1-p$ are called Bernoulli trials. Let P_n denote the probability that n Bernoulli trails result in an even number of successes (0 being considered an even number). Show that

$$P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1} \qquad n \geq 1$$

and use this to prove (by induction) that

$$P_n = \frac{1 + (1 - 2p)^n}{2}$$

Solution: Consider the outcome of the first toss: let A be the event of a success at the first trial, and B be the event that the remaining $n-1$ trials produce an even number of successes. Then A and B are independent (as the trials are) and

$$P_n = P(A\cap\overline{B})+P(\overline{A}\cap B) = P(A)P(\overline{B})+P(\overline{A})P(B) = p(1-P_{n-1})+(1-p)P_{n-1}$$

So we have proven that $P_n = p + (1 - 2p)P_{n-1}$.

Now $P_0 = 1$ and the induction proof is straight forward.

Review problems.

- **Problem.** Prove that if $P(A) = P(B) = \frac{3}{4}$, then $P(A|B) \geq \frac{2}{3}$.

Hint: Use the inclusion-exclusion formula.

Solution:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) + P(B) - P(A \cup B)}{P(B)}$$

$$= 2 - \frac{P(A \cup B)}{3/4} \geq 2 - \frac{1}{3/4} = 2 - \frac{4}{3} = \frac{2}{3}$$

as $\frac{P(A \cup B)}{3/4} \leq \frac{1}{3/4}$

Review problems.

• **Problem.** Let $S = \{a, b, c\}$ be the sample space for the experiment with positive probabilities for each outcome. Given three events $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, and $B = \{b\}$. Check if the following is true regardless of the probability values for the outcomes:

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$$

Solution: $A_1 \cup A_2 = S = \{a, b, c\}$

Hence,

$$P(A_1 \cup A_2|B) = P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

While,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

and

$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Hence $P(A_1 \cup A_2|B) \neq P(A_1|B) + P(A_2|B)$

Review problems.

• **Problem.** Consider two urns, one containing 1 black and 7 red marble, the other containing 6 black and 1 red marble. An urn is selected at random, and a marble is drawn at random from the selected urn. What is the probability that the first urn was the one selected, given that the marble is red?
Hint: use Bayes' formula, $P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F)+P(E|\overline{F})P(\overline{F})}$

Solution: Let $U_1 = \{\text{Urn 1 selected}\}$, $U_2 = \{\text{Urn 2 selected}\}$, and $R = \{\text{red marble selected}\}$. Then

$$P(R) = P(R \cap U_1) + P(R \cap U_2) = P(R|U_1)P(U_1) + P(R|U_2)P(U_2) = \frac{7}{8} \cdot \frac{1}{2} + \frac{1}{7} \cdot \frac{1}{2} = \frac{57}{112}$$

$$P(U_1|R) = \frac{P(R \cap U_1)}{P(R)} = \frac{P(R|U_1)P(U_1)}{P(R|U_1)P(U_1) + P(R|U_2)P(U_2)} = \frac{7/16}{57/112} = \frac{49}{57}$$

Review problems.

• **Problem.** An infinite sequence of independent Bernoulli trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. Denote by $s(n)$ the number of successes and by $f(n)$ the number of failure in the first n trials. For all integer $k > 0$, let $n = 2k$ and let A_n be the event that $s(n) - f(n) = 2$. Find the probability $P(A_n)$.

Solution:

$s(n) - f(n) = 2$ and $s(n) + f(n) = n = 2k$ if and only if

$$s(n) = k + 1 \quad \text{and} \quad f(n) = k - 1$$

Therefore $P(A_n)$ is the probability of $k + 1$ successes out of $n = 2k$ trials, and therefore is equal to

$$P(A_n) = \binom{2k}{k+1} p^{k+1}(1-p)^{k-1}$$

Review problems.

- **Problem.** Let A_n $(n = 2k > 0)$ be defined as in the preceding problem. Find $P(A_4 \cap A_6)$.

Solution: $P(A_4 \cap A_6) = P(A_6|A_4)\, P(A_4)$, where

$$P(A_4) = \binom{4}{3}\, p^3(1-p)$$

and $P(A_6|A_4)$ is the probability of exactly one success out of two trials, and therefore,

$$P(A_6|A_4) = 2p(1-p).$$

Thus, $P(A_4 \cap A_6) = P(A_6|A_4)\, P(A_4) = 8p^4(1-p)^2$.