MTH 463/563

Lectures 4 - 7

Yevgeniy Kovchegov
Oregon State University

Topics:

- Sets. Venn diagrams. De Morgan's laws
- Propositional calculus and events.
- Axioms of probability.
- Probability by counting.
- Properties of probability function.
- Inclusion-exclusion formula of probability.

Sets: notions and examples.

- a set is a collection of objects (elements).

- Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

- Rational numbers:

$\mathbb{Q} = \{\frac{n}{m} :\ n$ and m are integers, and $m \neq 0\}$

- Real numbers:

$\mathbb{R} = \{$ all values between $-\infty$ and $+\infty\}$

- Empty set $\emptyset = \{\}$

- U is called a universal set or a universe.
- $a \in A$ denotes that a is an element of A.
- $\overline{A}$ = all elements in the universe U that do not belong to A. Other notation: A^c.
- Intersection: $A \cap B$ = all elements in the universe U that belong to A **and** B. Other notation: AB.
- Union: $A \cup B$ = all elements in the universe U that belong to A **or** B, or to both sets, A and B.
- $A \setminus B$ = all elements in the universe U that belong to A but do **not** belong to B. Other notation: $A - B$.
- $A \subseteq B$ (A is a subset of B), i.e., all elements in A also belong to B.

Sets: notions and examples

Example.

Let the universe be the set of all digits

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{2, 3, 5, 7, 9\}$, and $E = \{0, 2, 4, 6, 8\}$. Then

- $\overline{A} = U - A = \{7, 8, 9\}$
- $A \cup E = \{0, 1, 2, 3, 4, 5, 6, 8\}$
- $A \cap E = \{0, 2, 4, 6\}$
- $A - B = \{0, 1, 4, 6\}$

Sets: notions and examples

Example.

Let the universe be the set of all digits

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{2, 3, 5, 7, 9\}$, and $E = \{0, 2, 4, 6, 8\}$. Then

- $\overline{A \cap E} \cap B = \{1, 3, 5, 7, 8, 9\} \cap B = \{3, 5, 7, 9\}$
- $\overline{A \cup E} \cap B = \{7, 9\} \cap B = \{7, 9\}$
- $A \cap (B \cup E) = A \cap \{0, 2, 3, 4, 5, 6, 7, 8, 9\} = \{0, 2, 3, 4, 5, 6\}$
- $(A \cap B) \cup E = \{2, 3, 5\} \cup E = \{0, 2, 3, 4, 5, 6, 8\}$

Sets: notions and examples

Example.

Let the universe be the set of all digits

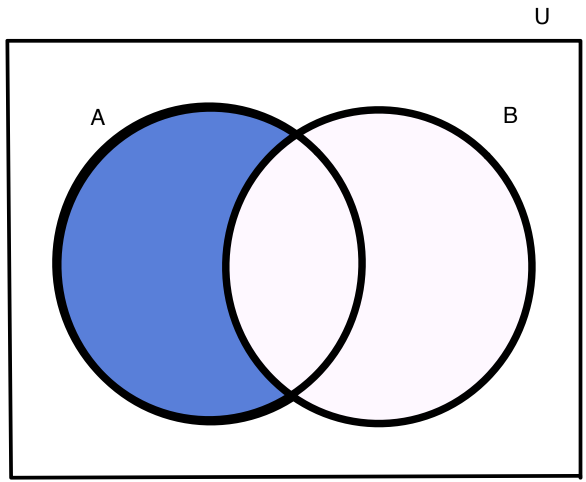
$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{2, 3, 5, 7, 9\}$, and $E = \{0, 2, 4, 6, 8\}$. Then

- $A - (B \cup E) = A - \{0, 2, 3, 4, 5, 6, 7, 8, 9\} = \{1\}$
- $(B \cup E) - A = \{0, 2, 3, 4, 5, 6, 7, 8, 9\} - A = \{7, 8, 9\}$
- $(B \cap E) - A = \{2\} - A = \emptyset$

Venn diagrams

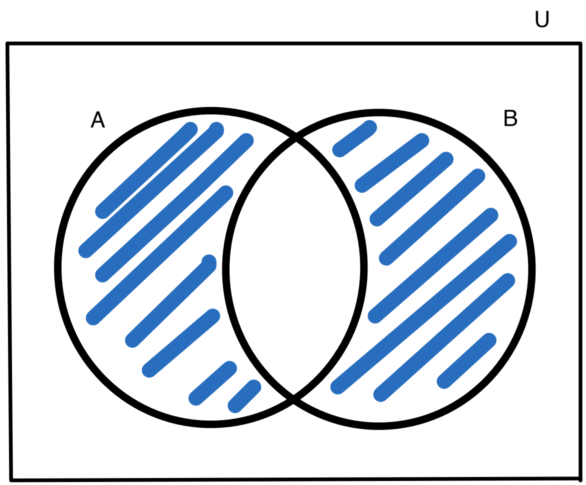
- Shade $A \cap \overline{B}$



$A \cap \overline{B} = A - B$ represents all elements in the universe U that belong to the set A, but do not belong to B.

Venn diagrams

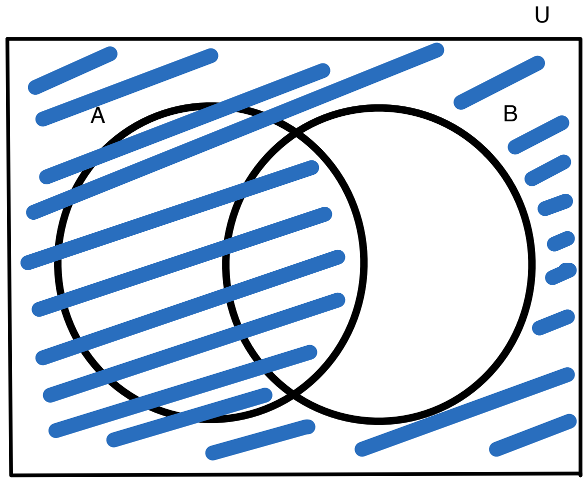
- Shade $(A \cup B) - (A \cap B)$



$(A \cup B) - (A \cap B)$ represents all elements in the universe U that belong to A, **or** B, but do not belong to both A **and** B.

Venn diagrams

- Shade $A \cup \overline{B}$

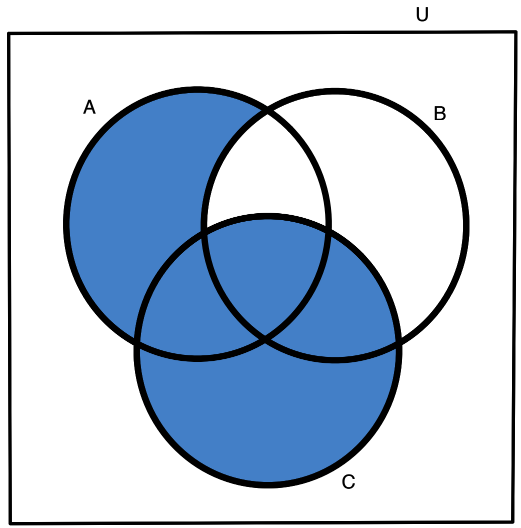


$A \cup \overline{B} = \overline{(B - A)}$ represents all elements in the universe U that belong to A, **or** do not belong to B, **or** both belong to A and do not belong to B.

Venn diagrams

- Shade $(A \cap \overline{B}) \cup C$

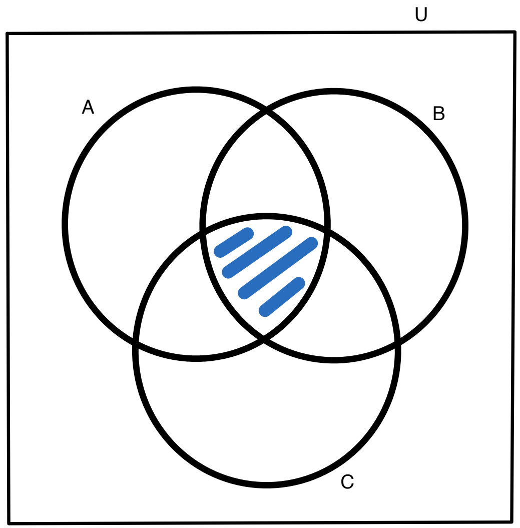
Here $(A \cap \overline{B}) \cup C = (A - B) \cup C$ represents all elements in the universe U that belong to C, **or** that belong to A and do not belong to B.



Venn diagrams

- Shade $A \cap B \cap C$

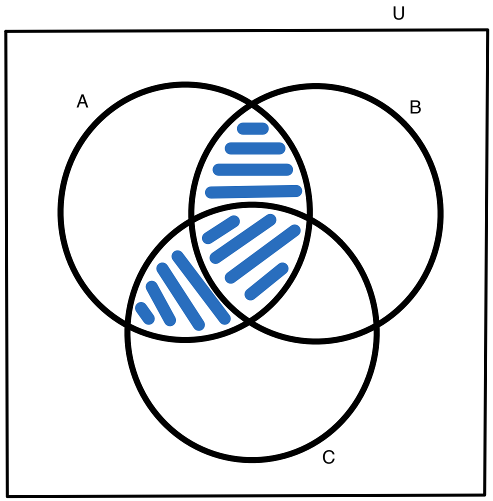
Here $A \cap B \cap C$ represents all elements in the universe U that belong to A **and** B **and** C altogether.



Venn diagrams

- Shade $A \cap (B \cup C)$

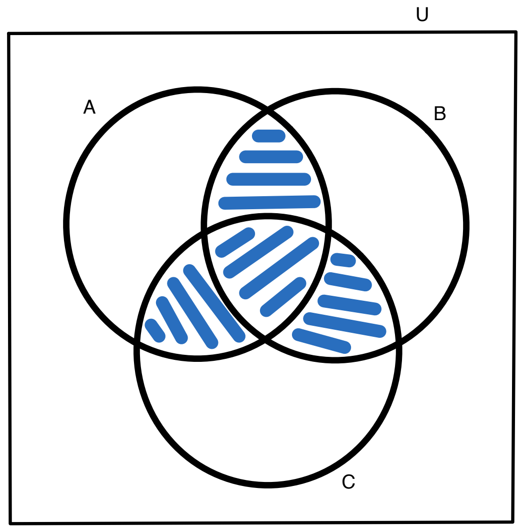
Here $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ represents all elements in the universe U that belong to A, **and** to at least one of the two other sets, B **or** B.



Venn diagrams

- Shade $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

Here $(A \cap B) \cup (A \cap C) \cup (B \cap C)$ represents all elements in the universe U that belong to at least two of the three sets, A, B and C.



Rules of set theory:

- Commutative laws:

$$E \cup F = F \cup E \qquad E \cap F = F \cap E$$

- Associative laws:

$$(E \cup F) \cup G = E \cup (F \cup G) \qquad (E \cap F) \cap G = E \cap (F \cap G)$$

- Distributive laws:

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$$

$$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$$

Notations:

$$\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \ldots \cap A_n$$

Example. $\bigcup_{j=1}^{3} A_j = A_1 \cup A_2 \cup A_3$

Example. $\bigcap_{j=1}^{5} A_j = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$

De Morgan's laws:

Consider sets $E_1,\ E_2, \ldots, E_n$, then

$$\overline{\left(\bigcup_{j=1}^{n} E_j\right)} = \bigcap_{j=1}^{n} \overline{E}_j$$

and

$$\overline{\left(\bigcap_{j=1}^{n} E_j\right)} = \bigcup_{j=1}^{n} \overline{E}_j$$

De Morgan's laws:

$$\overline{\left(\bigcup_{j=1}^{n} E_j\right)} = \bigcap_{j=1}^{n} \overline{E}_j$$

Proof:

$$x \in \overline{\left(\bigcup_{j=1}^{n} E_j\right)} \quad \Leftrightarrow \quad x \notin \bigcup_{j=1}^{n} E_j \quad \Leftrightarrow \quad x \notin E_j \text{ for all } j = 1, \ldots, n$$

$$\Leftrightarrow \quad x \in \overline{E}_j \text{ for all } j = 1, \ldots, n \quad \Leftrightarrow \quad x \in \bigcap_{j=1}^{n} \overline{E}_j$$

Hence $\overline{\left(\bigcup_{j=1}^{n} E_j\right)} = \bigcap_{j=1}^{n} \overline{E}_j$

Propositional calculus

Consider propositions, p and q. Then

- $\neg p$ is a proposition **“ not p ”**
- $p \wedge q$ is a proposition **“ p and q ”**
- $p \vee q$ is a proposition **“ p or q ”** meaning “p or q, or both”
- $p \rightarrow q$ (**conditional implication**) means that the truth of p implies the truth of q

Propositional calculus and sets

$\neg p$ corresponds to $\overline{A}$: $(x \in \overline{A}) \Leftrightarrow \neg(x \in A)$

$p \wedge q$ corresponds to $A \cap B$: $(x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)$

$p \vee q$ corresponds to $A \cup B$: $(x \in A \cup B) \Leftrightarrow (x \in A) \vee (x \in B)$

and

$p \rightarrow q$ corresponds to $A \subseteq B$: $A \subseteq B \Leftrightarrow (x \in A) \rightarrow (x \in B)$

Finally, the **truth tables** are similar to Venn diagrams.

Truth thabels

- Given proposition p, here is the truth table for $\neg p$

p	$\neg p$
T	F
F	T

- Given propositions p and q, here is the truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Given propositions p and q, here is the truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Introduction to discrete probability.

• Sample space $\mathcal{S}$ = the space of all possible outcomes.

• Event = a set in the sample space $\mathcal{S}$.

• Let $E \subseteq \mathcal{S}$ be an event. Its complement $\overline{E} = S \setminus E$ is the event not E

• $E \cap F$ = event E **and** F

• $E \cup F$ = event E **or** F

• Events E and F are said to be mutually exclusive or disjoint if

$$E \cap F = \emptyset$$

Axioms of probability.

- $0 \leq P(E) \leq 1$ for any event $E \subseteq \mathcal{S}$
- $P(\mathcal{S}) = 1$
- If $E_1, E_2, \ldots$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$$

Probability by counting.

• **Example.** A fair die is the one producing each outcome with equal probability. If we roll a fair die, the sample space will consist of six outcome

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}, \qquad \text{with probabilities}$$

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Let $E = \{1, 2\}$ be the event of a low score (i.e. 2 or lower). Its probability will be

$$P(E) = P(1) + P(2) = \frac{2}{6} = \frac{1}{3}$$

Here, since every outcome in the sample space is equally likely,

$$P(A) = \frac{|A|}{|\mathcal{S}|} \qquad \text{for each event} \quad A \subseteq \mathcal{S}$$

Probability by counting.

In general, **in case when every outcome in the sample space $\mathcal{S}$ is equally likely,**

$$P(A) = \frac{|A|}{|\mathcal{S}|} \quad \text{for every event} \quad A \subseteq \mathcal{S},$$

where $|A|$ denotes the number of elements in A.

• **Example.** Roll two fair dice. Find the probability of the sum being divisible by three.

Here $\mathcal{S} =$

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Let $E = \{\text{the sum is divisible by three}\}$. Then

$$P(E) = \frac{|E|}{|\mathcal{S}|} = \frac{12}{36} = \frac{1}{3}$$

Probability by counting.

• **Example.** Toss a fair coin three times. There each outcome in the sample space

$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

is equally likely. We need to find the probability of one head (H) and two tails (T) in the outcome of the three tosses.

Let the event $E = \{\text{one H and two T}\}$:

$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Since in this example, every outcome in $\mathcal{S}$ is equally likely,

$$P(E) = \frac{|E|}{|\mathcal{S}|} = \frac{\binom{3}{1}}{2^3} = \frac{3}{8}$$

Probability by counting.

- **Example.** Toss a fair coin n times. There each outcome in the sample space

$$\mathcal{S} = \{n\text{-long words of H and T}\}$$

is equally likely. We need to find the probability of k heads (H) and $n-k$ tails (T) in the outcome of n tosses.

Let the event

$$E = \{n\text{-long words of H and T with } k \text{ Hs and } n-k \text{ Ts}\}$$

Since in this example, every outcome in $\mathcal{S}$ is equally likely,

$$P(E) = \frac{|E|}{|\mathcal{S}|} = \frac{\binom{n}{k}}{2^n}$$

Probability by counting.

• **Example.** A deck of 52 cards is dealt among four players. What is the probability that each player gets exactly one ace?

• **Solution.** Here the cards can be dealt in

$$|\mathcal{S}| = \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!} = \frac{52!}{(13!)^4}$$

different ways, each outcome being equally likely.

Let $E = \{\text{each player gets one ace}\}$. Next, we find the number $|E|$ of the outcomes in E.

We deal the cards in **two** steps. First, we distribute the four aces among the four player so that each gets exactly one ace. We can do it in 4! ways. Second, we distribute the rest 48 cards to the four players, each getting 12 cards. This can be done in

$$\frac{48!}{12! \cdot 12! \cdot 12! \cdot 12!} = \frac{48!}{(12!)^4}$$

ways. Thus, by the multiplicative rule of counting, $|E| = 4! \cdot \frac{48!}{(12!)^4}$.

Probability by counting.

• **Example.** A deck of 52 cards is dealt among four players. What is the probability that each player gets exactly one ace?

• **Solution (continued).** We have

$$|\mathcal{S}| = \frac{52!}{(13!)^4} \quad \text{and} \quad |E| = 4! \cdot \frac{48!}{(12!)^4}$$

Now, since every outcome in $\mathcal{S}$ is equally likely,

$$P(E) = \frac{|E|}{|\mathcal{S}|} = \frac{4! \cdot 48!}{52!} \cdot (13)^4 \approx 0.1055$$

Probability by counting.

• **Example.** An urn contains G green and B blue balls. If a random sample of size n is chosen, what is the probability that it contains k green balls?

• **Solution:** We answer the question by counting the number of ways to select k green and $n-k$ blue balls, and then dividing by the total number of possible selections:

$$\frac{\binom{G}{k}\binom{B}{n-k}}{\binom{G+B}{n}}$$

Examples.

• **Example.** An urn contains n green and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that with probability $\frac{n}{n+m}$ they are all green.

Solution: The outcome of the experiment will not change if you withdraw out all but one last marble from the urn, and then check its color. Now, removing $n+m-1$ marbles from the urn, and checking the color of the last marble is no different from selecting one marble and checking its color. Both are equivalent to separating marbles into two groups, one of size one, and the other of size $n+m-1$. Thus the marble is green with probability $\frac{n}{n+m}$.

Axioms of probability.

- $0 \leq P(E) \leq 1$ for any event $E \subseteq \mathcal{S}$

- $P(\mathcal{S}) = 1$

- If $E_1, E_2, \ldots$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$$

Properties of probability function.

- **Proposition.** $P(\emptyset) = 0$ ($\emptyset$ - the empty set).
- **Proposition.** $P(\overline{E}) = 1 - P(E)$
- **Lemma.** If $E \subseteq F$, then
$$P(F) = P(E) + P(F \cap \overline{E})$$
- **Proposition.** If $E \subseteq F$, then $P(E) \leq P(F)$
- **Inclusion-Exclusion Theorem.**
$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Properties of probability function.

- **Proposition.** $P(\emptyset) = 0$ ($\emptyset$ - the empty set).
- **Proof:** The third axiom of probability states:

If $E_1, E_2, \ldots$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$$

Here $\mathcal{S} \cap \emptyset = \emptyset$, and therefore

$$P(\mathcal{S}) = P(\mathcal{S} \cup \emptyset \cup \emptyset \ldots) = P(\mathcal{S}) + P(\emptyset) + P(\emptyset) + \ldots$$

Hence $P(\emptyset) = 0$.

Properties of probability function.

- **Proposition.** $P(\overline{E}) = 1 - P(E)$
- **Proof:** The third axiom of probability implies:

If $E_1 \cap E_2 = \emptyset$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Here $E \cap \overline{E} = \emptyset$, and therefore

$$P(\mathcal{S}) = P(E \cup \overline{E}) = P(E) + P(\overline{E}),$$

where $P(\mathcal{S}) = 1$ by the second axiom of probability.

Hence $P(E) + P(\overline{E}) = 1$, implying $P(\overline{E}) = 1 - P(E)$.

Properties of probability function.

- **Proposition.** For any two events A and B in $\mathcal{S}$, $P(A) = P(A \cap B) + P(A \cap \overline{B})$.

- **Proof:** Since $A \subseteq \mathcal{S}$,

$$A = A \cap \mathcal{S} = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$$

by the distributivity law of set theory.

Thus $A = (A \cap B) \cup (A \cap \overline{B})$, where $A \cap B$ is a subset of B and $A \cap \overline{B}$ is a subset of $\overline{B}$.

Therefore $(A \cap B) \cap (A \cap \overline{B}) = \emptyset$, and

$$P(A) = P\big((A \cap B) \cup (A \cap \overline{B})\big) = P(A \cap B) + P(A \cap \overline{B})$$

by the third axiom of probability.

Properties of probability function.

Lemma. If $E \subseteq F$, then

$$P(F) = P(E) + P(F \cap \overline{E})$$

- **Proof:** The preceding proposition states that

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

Now, since $E \subseteq F$, $E \cap F = E$ and therefore

$$P(F) = P(F \cap E) + P(F \cap \overline{E}) = P(E) + P(F \cap \overline{E})$$

Properties of probability function.

• **Proposition.** If $E \subseteq F$, then $P(E) \leq P(F)$

• **Proof:** The preceding lemma states that if $E \subseteq F$, then

$$P(F) = P(E) + P(F \cap \overline{E}),$$

where $P(F \cap \overline{E}) \geq 0$ by the first axiom of probability.

Hence

$$P(F) = P(E) + P(F \cap \overline{E}) \geq P(E) + 0$$

Examples.

- **Example.** Show that the probability that exactly one of the two events, E or F, occurs is

$$P(E) + P(F) - 2P(E \cap F)$$

Solution: $(E \cap \overline{F}) \cup (\overline{E} \cap F)$ is the event that exactly one of the two occurs. The inclusion-exclusion formula implies

$$P\big((E\cap\overline{F})\cup(\overline{E}\cap F)\big) = P(E\cap\overline{F})+P(\overline{E}\cap F) = P(E)+P(F)-2P(E\cap F)$$

as

$$P(E \cap \overline{F}) = P(E) - P(E \cap F)$$

and

$$P(\overline{E} \cap F) = P(F) - P(E \cap F)$$

Inclusion-Exclusion Theorem.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Proof:** We proved the following lemma:

If $E \subseteq F$, then $P(F) = P(E) + P(F \cap \overline{E})$.

Taking $E = A$ and $F = A \cup B$, we obtain

$$P(A \cup B) = P(A) + P(B \cap \overline{A})$$

as $F \cap \overline{E} = (A \cup B) \cap \overline{A} = (A \cap \overline{A}) \cup (B \cap \overline{A}) = \emptyset \cup (B \cap \overline{A}) = B \cap \overline{A}$

Also we proved the following proposition:

$$P(B) = P(B \cap A) + P(B \cap \overline{A})$$

Hence $P(B \cap \overline{A}) = P(B) - P(B \cap A)$ and

$P(A \cup B) = P(A) + P(B \cap \overline{A}) = P(A) + P(B) - P(A \cap B)$

Inclusion-Exclusion Theorem.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

- **Example.** Roll two fair dice.

Here $\mathcal{S} =$

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Let $E = \{$the sum is divisible by three$\}$ (in orange),
and $F = \{$ the sum is $\geq$ 9 $\}$ (blue shading).

Then $P(E) = \dfrac{|E|}{|\mathcal{S}|} = \dfrac{12}{36} = \dfrac{1}{3}$, $P(F) = \dfrac{|F|}{|\mathcal{S}|} = \dfrac{10}{36} = \dfrac{5}{18}$, and $P(E \cap F) = \dfrac{|E \cap F|}{|\mathcal{S}|} = \dfrac{5}{36}$

Then $P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{12}{36} + \frac{10}{36} - \frac{5}{36} = \frac{17}{36}$

Inclusion-Exclusion Theorem.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

- **General case:**

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_r})$$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \ldots + (-1)^{n+1} P(E_1 \cap E_2 \cap \ldots \cap E_n)$$

- **Example.**

$$P(E_1 \cup E_2 \cup E_3) = \sum_{i=1}^{3} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

Bonferroni's inequality

- **Example.** Let us prove Bonferroni's inequality:

$$P(E \cap F) \geq P(E) + P(F) - 1$$

Solution: The Inclusion-Exclusion Theorem implies

$$P(E \cap F) = P(E) + P(F) - P(E \cup F) \geq P(E) + P(F) - 1$$

Comment: Thus we can show that if $P(E) = 0.85$ and $P(F) = 0.75$, then $P(E \cap F) \geq 0.6$.

Inclusion-Exclusion Theorem.

- **Example.**

$$P(E_1 \cup E_2 \cup E_3) = \sum_{i=1}^{3} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1)+P(E_2)+P(E_3)-P(E_1 \cap E_2)-P(E_1 \cap E_3)-P(E_2 \cap E_3)+P(E_1 \cap E_2 \cap E_3)$$

Proof:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P\big((E_1 \cup E_2) \cup E_3\big) = P(E_1 \cup E_2) + P(E_3) - P\big((E_1 \cup E_2) \cap E_3\big) \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) + P(E_3) - P\big((E_1 \cup E_2) \cap E_3\big) \\ &= P(E_1)+P(E_2)+P(E_3)-P(E_1 \cap E_2)-P\big((E_1 \cap E_3) \cup (E_2 \cap E_3)\big) \end{aligned}$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

as

$$P\big((E_1 \cap E_3) \cup (E_2 \cap E_3)\big) = P(E_1 \cap E_3) + P(E_2 \cap E_3) - P\big((E_1 \cap E_3) \cap (E_2 \cap E_3)\big)$$