MTH 463/563

Lectures 1-3

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Topics:

- Counting: Multiplicative rule.

- Permutations.

- Combinations.

- Generalized combinations.

- Binomial theorem. Multinomial theorem.

- Counting solutions of $x_1 + \cdots + x_k = n$

- More combinatorial identities.

COMBINATORICS

Counting: Multiplicative rule

Two experiments are being performed.

• Experiment 1:   Tossing a coin. Possible outcomes are H (heads) and T (tails). The number of outcomes $n_1 = 2$.

• Experiment 2: Rolling a die. Possible outcomes are 1,2,3,4,5 and 6. The number of outcomes $n_2 = 6$.

The total number of possible outcomes is

$$n_1 \times n_2 = 12$$

They are

(H,1), (H,2), (H,3), (H,4), (H,5), (H,6),

(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)

Counting: Multiplicative rule

Similarly, if k experiments are being performed.

• Experiment 1: n_1 possible outcomes

• Experiment 2: n_2 possible outcomes (per each outcome from experiment 1)

• Experiment 3: n_3 possible outcomes (per each outcome from experiments 1 and 2)

$$\vdots$$

• Experiment k: n_k possible outcomes (per each outcome from experiments 1 through $k - 1$)

Then the total number of possible outcomes is

$$n_1 \times n_2 \times \cdots \times n_k$$

Permutations

Question: In how many ways can n distinct objects be ordered?

Examples:

• The order in which n runners finish a race.

• The number of five letter strings $(n = 5)$ one can create with five letters, A, B, C, D and E, if each letter is used exactly once (BADCE is an example of such string).

• The number of distinct strings one can create with n distinct symbols, $A_1, A_2, \ldots, A_n$, if each symbol is used exactly once.

• The number of ways a deck of n distinct cards can be ordered (card shuffling)

• The number of ways n distinct people can be ordered in a line.

Each ordering of n distinct objects is called a **permutation**. We want to find the total number of possible permutations of n distinct objects.

Permutations: Counting all n-permutations can be done via n experiments.

Example. *Count the number of five letter strings one can create with five letters, A, B, C, D and E, if each letter is used exactly once.*

We have $n = 5$ spots to fill: _ _ _ _ _

• Experiment 1: Placing A. There are five available spots, hence there are $n_1 = 5$ possible outcomes of the experiment.

After we place A there are still four vacant spots in the string:

$$\text{\_ \_ \_ A \_}$$

- **Experiment 2:** Placing B. There are four available spots, hence there are $n_2 = 4$ possible outcomes of the experiment.

After we place B there are still three vacant spots in the string:

$$B \_ \_ A \_$$

- **Experiment 3:** Placing C. There are $n_3 = 3$ possible outcomes of the experiment.
After we place C there are still two vacant spots in the string:

$$B \_ C A \_$$

• **Experiment 4:** Placing D. There are $n_4 = 2$ possible outcomes of the experiment.
After we place D there are still one vacant spots in the string:

$$B \_ C A D$$

• **Experiment 5:** Placing E. There are $n_5 = 1$ possible outcomes of the experiment.
After we place E there are still no vacant spots in the string:

$$B E C A D$$

By the multiplicative rule,

$$n_1 \times n_2 \times n_3 \times n_4 \times n_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

is the total number of possible outcomes.

Observe that each outcome is a distinct string, and that each permutation is a possible outcome. Thus we counted all permutations of 5 distinct objects:

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

Permutations: Counting all permutations of n distinct objects can be done via n experiments. Therefore the multiplicative rule implies the following.

Proposition. The total number of permutations of n distinct objects is

$$n! = 1 \times 2 \times \cdots \times n$$

called n **factorial**.

Here, $1! = 1$, $2! = 1 \times 2 = 2$,
$3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$,
$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ and so on.
Also we let $0! = 1$.

Combinations.

Question: How many n-long strings is possible to create with k A's and $n - k$ B's ?

Answer:

$$\binom{n}{k} = \frac{n!}{k!(n - k)!}$$

Another way to denote $\binom{n}{k}$ is $C(n, k)$ (reads "n choose k").

Other alternative notations: C_n^k and $\ _nC_k$

Combinations.

Example: How many strings of length 8 is it possible to create with 5 As and 3 Bs (such as $BAAAABBA$)?

Answer:

$$\binom{8}{5} = \frac{8!}{5!3!} = 56$$

Why $\frac{8!}{5!3!}$**?** Let's index the 5 As and 3 Bs as follows

$$A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3$$

So with the 8 **different** characters we can make 8! different strings such as $B_2 A_3 A_5 A_2 A_1 B_1 B_3 A_4$

Alternatively, we can count as follows:

(1) Arranging 5 As and 3 Bs into an eight letter string such as $BAAAABBA$. There are $\binom{8}{5}$ ways.

(2) Ordering five As as in $BA_3 A_5 A_2 A_1 BBA_4$. This can be done in 5! different ways.

(3) Ordering three Bs as in $B_2 A_3 A_5 A_2 A_1 B_1 B_3 A_4$. This can be done in 3! different ways.

$$\text{Therefore,} \quad 8! = \binom{8}{5} 5! 3! \quad \text{and} \quad \binom{8}{5} = \frac{8!}{5!3!}$$

Why $\binom{n}{k} = \frac{n!}{k!(n-k)!}$**?** Let's index k letters A and $n-k$ letters B as follows

$$A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_{n-k}$$

With this n **different** characters we can make $n!$ different permutations.

Alternatively, we can count as follows:

(1) Arranging k letters A and $n-k$ letters B into an n letter string can be done in $\binom{n}{k}$ different ways.

(2) Ordering k letters A can be done in $k!$ different ways.

(3) Ordering $n-k$ letters B can be done in $(n-k)!$ different ways.

Hence, $n! = \binom{n}{k} k! \, (n-k)!$,   and therefore, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Properties of $\binom{n}{k}$.

- $\binom{n}{k} = \binom{n}{n-k}$

- $\binom{n}{0} = \binom{n}{n} = 1$

- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $0 < k < n$

Combinatorial proof: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Recall that $\binom{n}{k}$ is the number of distinct n-long words one can create with k letters A and $n-k$ letters B. Let us count them as follows.

• First let's count the words that begin with an A. There the first spot is occupied

$$A \_ \_ \_ \cdots \_ \_$$

and we need to fill in the remaining $n-1$ spots with $(k-1)$ As and $(n-k)$ Bs. Thus, there $\binom{n-1}{k-1}$ of them.

• Then let's count the words that begin with a B. There the first spot is also occupied

$$B \_ \_ \_ \cdots \_ \_$$

and we need to fill in the remaining $n-1$ spots with k As and $(n-k-1)$ Bs. Thus, there $\binom{n-1}{k}$ of them.

Algebraic proof: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{n-k}{n} \cdot \frac{n!}{k!(n-k)!} + \frac{k}{n} \cdot \frac{n!}{k!(n-k)!}$$

as $(n-1)! = n!/n$, $(k-1)! = k!/k$, and $(n-k-1)! = (n-k)!/(n-k)$

Thus,

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \left(\frac{n-k}{n} + \frac{k}{n}\right) \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Pascal's triangle. $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = ? \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = ? \qquad \binom{3}{2} = ? \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = ? \qquad \binom{4}{2} = ? \qquad \binom{4}{3} = ? \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = ? \qquad \binom{3}{2} = ? \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = ? \qquad \binom{4}{2} = ? \qquad \binom{4}{3} = ? \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = ? \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = ? \qquad \binom{4}{2} = ? \qquad \binom{4}{3} = ? \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

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$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = ? \qquad \binom{4}{2} = ? \qquad \binom{4}{3} = ? \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = \text{?} \qquad \binom{4}{3} = \text{?} \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = ? \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Pascal's triangle. $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Generalized combinations.

Question: How many n-long strings is possible to create with

k_1 A's, k_2 B's   and $n - k_1 - k_2$ C's ?

Answer:

$$\binom{n}{k_1 + k_2} \cdot \binom{k_1 + k_2}{k_1} = \frac{n!}{k_1! k_2! (n - k_1 - k_2)!}$$

Proof: There are two experiments.
(i) Placing $k = k_1 + k_2$ Œs and $n - k$ Cs in the n-long string:

$$\text{Œ Œ } C \text{ Œ } C \; C \; C \text{ Œ Œ } C \; C$$

This can be done in $\binom{n}{k}$ ways.

(ii) Replacing Œs with k_1 As and k_2 Bs:

$$A \; B \; C \; A \; C \; C \; C \; A \; B \; C \; C$$

This can be done in $\binom{k}{k_1}$ ways.

Answer:

$$\binom{n}{k} \cdot \binom{k}{k_1} = \frac{n!}{k_1! k_2! (n - k_1 - k_2)!}$$

Generalized combinations.

Example: How many 24-long strings is possible to create with

7 A's,   11 B's   and   6 C's ?

(e.g. BCBBCABAACCCBBBAABBBBACA)

Answer:

$$\binom{24}{18} \cdot \binom{18}{7} = \frac{24!}{7! \cdot 11! \cdot 6!}$$

Generalized combinations.

Question: Count the number of different ways in which n objects of r different types can be ordered if there are

k_1 objects of type 1,
k_2 objects of type 2,
$\ldots$
k_r objects of type r.

All in all the total of $k_1 + k_2 + \cdots + k_r = n$ objects.

Answer:

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- **Example.** For $n = 2$,

$$(x+y)^2 = \binom{2}{0} x^0 y^2 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^2 y^0$$

$$= y^2 + 2xy + x^2$$

Binomial Theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- **Example.** For $n = 3$,

$$(x+y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0$$

$$= y^3 + 3xy^2 + 3x^2 y + x^3$$

Binomial Theorem: $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

● **Proof.** The Binomial theorem follows easily from the definition of $\binom{n}{k}$.
Take $n = 2$,

$$(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy$$

Observe that we sum up over all possible 2-long strings that can be created with x and y,

$$xx, \ xy, \ yx, \ \text{and} \ \ yy$$

● There are $\binom{2}{2} = 1$ strings with two x and no y
● There are $\binom{2}{1} = 2$ strings with one x and one y
● There are $\binom{2}{0} = 1$ strings with no x and two y

We consolidate the terms to obtaining

$$(x + y)^2 = x^2 + 2xy + y^2$$

Take $n = 3$,

$$(x+y)^3 = (x+y)(x+y)(x+y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

Observe that we sum up over all possible 3-long strings that can be created with x and y.

- There are $\binom{3}{3} = 1$ strings with three x and no y
- There are $\binom{3}{2} = 3$ strings with two x and one y
- There are $\binom{3}{1} = 3$ strings with one x and two y
- There are $\binom{3}{0} = 1$ strings with no x and three y

We consolidate the terms to obtaining

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

When we factor $(x + y)^n$, each $(x + y)$ contributes either x or y to the resulting n-long string:

$$(x + y)(x + y)(x + y) \ldots (x + y)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \cdots \qquad \downarrow$$

$$x \text{ or } y \quad x \text{ or } y \quad x \text{ or } y \quad \ldots \quad x \text{ or } y$$

Multiplying out $(x+y)(x+y)(x+y)\ldots(x+y)$ we end up with all possible distinct n-long strings made of x and y.

We consolidate the terms that have the same number of x and y:

• There are $\binom{n}{k}$ strings with k letters x and $n - k$ letters y.

Hence $(x + y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Binomial Theorem: $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

This is why the numbers $\binom{n}{k}$ are also known as **binomial coefficients**.

• **Example.** Find the coefficient in front of $x^7 y^9$ in the expansion of $(x + y)^{16}$.

• **Answer:** $\binom{16}{7} = 11,440$

Binomial Theorem: $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

- **Example.** Find $\sum\limits_{k=0}^{n} \binom{n}{k}$.

- **Solution:** Observe that if we use the Binomial Thoerem with $x = y = 1$, the left hand side $\sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ becomes $\sum\limits_{k=0}^{n} \binom{n}{k}$.

Thus

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n$$

Generalized combinations.

Question: Count the number of different ways in which n objects of r different types can be ordered if there are

k_1 objects of type 1,
k_2 objects of type 2,
$\ldots$
k_r objects of type r.

All in all the total of $k_1 + k_2 + \cdots + k_r = n$ objects.

Answer:

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}$$

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{k_1 \geq 0, \ldots, k_r \geq 0 \\ k_1 + k_2 + \cdots + k_r = n}} \frac{n!}{k_1! \cdot k_2! \cdot \cdots \cdot k_r!} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

Example:

$$
\begin{aligned}
(x + y + z)^2 &= \frac{2!}{2!0!0!} x^2 y^0 z^0 + \frac{2!}{0!2!0!} x^0 y^2 z^0 + \frac{2!}{0!0!2!} x^0 y^0 z^2 \\
&+ \frac{2!}{0!1!1!} x^0 y^1 z^1 + \frac{2!}{1!0!1!} x^1 y^0 z^1 + \frac{2!}{1!1!0!} x^1 y^1 z^0 \\
&= x^2 + y^2 + z^2 + 2yz + 2xz + 2xy
\end{aligned}
$$

Multinomial Theorem can be proved in the same way as the Binomial Theorem. Alternatively, it can be proved using the Binomial Theorem.

Question: How many strings can be formed with the following letters?

$$S\ H\ E\ S\ E\ L\ L\ S\ S\ E\ A\ S\ H\ E\ L\ L\ S$$

Solution: There are

$$k_1 = 1 \text{ letter A,}$$
$$k_2 = 4 \text{ letters E,}$$
$$k_3 = 2 \text{ letters H,}$$
$$k_4 = 4 \text{ letters L,}$$
$$k_5 = 6 \text{ letters S}$$

Therefore there must be

$$\frac{17!}{1!4!2!4!6!}$$

different strings made with these letters.

Question: In how many ways can 52 different cards be dealt between four players (call them Player A, Player B, Player C, and Player D) where each gets exactly 13 cards?

Solution: Line up the cards: $\boxed{1}$, $\boxed{2}$, . . . , $\boxed{52}$

Each card is assigned a player. Thus there will be 4 types of cards: the ones that belong to Player A, the ones that belong to Player B, the ones that belong to Player C, and the ones that belong to Player D. Hence there are $\frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}$ ways to deal the deck.

Counting solutions of $x_1 + \cdots + x_k = n$

Question: Count the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 15$$

subject to $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$, and $x_4 \geq 1$.

Solution: Represent the devision of 15 marbles into the four groups containing x_1, x_2, x_3, and x_4 marbles respectively, as follows.

$$\underbrace{ooo}_{x_1} \Big| \underbrace{oooo}_{x_2} \Big| \underbrace{oo}_{x_3} \Big| \underbrace{oooooo}_{x_4}$$

Now, there are 15-1=14 spaces between the marbles and we need to place 4-1=3 divisors | into three of the spaces. Two divisors cannot occupy the same space since each $x_j \geq 1$.

Counting solutions of $x_1 + \cdots + x_k = n$

$$\underbrace{\circ\circ\circ}_{x_1} \mid \underbrace{\circ\circ\circ\circ}_{x_2} \mid \underbrace{\circ\circ}_{x_3} \mid \underbrace{\circ\circ\circ\circ\circ\circ}_{x_4}$$

Thus there are $\binom{15-1}{4-1} = \binom{14}{3} = 364$ ways to split the 15 marbles.

In general, for $0 < k \le n$, there are $\binom{n-1}{k-1}$ integer solutions of

$$x_1 + x_2 + \cdots + x_k = n$$

subject to $x_1, x_2, \ldots, x_k$ being positive:

$$x_1 \ge 1, \quad x_2 \ge 1, \quad \ldots , x_n \ge 1$$

Counting solutions of $x_1 + \cdots + x_k = n$

Question: Count the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 15$$

subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, and $x_4 \geq 0$.

Solution: Representing the divisors is very difficult since now the two or three divisors may end up in the same space. Instead we use the previous result.

Let $y_1 = x_1 + 1$, $y_2 = x_2 + 1$, $y_3 = x_3 + 1$, and $y_4 = x_4 + 1$.

Then $x_1 + x_2 + x_3 + x_4 = 15$ subject to $x_1, x_2, x_3, x_4 \geq 0$

if and only if

$y_1 + y_2 + y_3 + y_4 = 19$ subject to $y_1, y_2, y_3, y_4 \geq 1$,

where 19 was obtained as $15 + 4$.

Counting solutions of $x_1 + \cdots + x_k = n$

So, $x_1 + x_2 + x_3 + x_4 = 15$ subject to $x_1,\ x_2,\ x_3,\ x_4\ \geq 0$
if and only if
$y_1 + y_2 + y_3 + y_4 = 19$ subject to $y_1,\ y_2,\ y_3,\ y_4\ \geq 1$,
where 19 was obtained as $15 + 4$.

The latter problem has $\binom{18}{3}$ solutions.

In general, using the same trick, we show that there
are $\binom{n+k-1}{k-1}$ integer solutions of

$$x_1 + x_2 + \cdots + x_k = n$$

subject to

$$x_1 \geq 0,\quad x_2 \geq 0,\quad \ldots\ , x_k \geq 0$$

More combinatorial identities.

Question: For $0 < k \le n$, find

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k}$$

Solution: Recall $\binom{m+1}{k+1} = \binom{m}{k} + \binom{m}{k+1}$. Thus $\binom{m}{k} = \binom{m+1}{k+1} - \binom{m}{k+1}$

$$\binom{k}{k} = \binom{k+1}{k+1} \qquad \binom{k+1}{k} = \binom{k+2}{k+1} - \binom{k+1}{k+1} \qquad \binom{k+2}{k} = \binom{k+3}{k+1} - \binom{k+2}{k+1}$$

$$\binom{k+3}{k} = \binom{k+4}{k+1} - \binom{k+3}{k+1} \quad \cdots \quad \binom{n-1}{k} = \binom{n}{k+1} - \binom{n-1}{k+1} \quad \binom{n}{k} = \binom{n+1}{k+1} - \binom{n}{k+1}$$

Adding together both sides of the above, we obtain

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{\mathbf{n+1}}{\mathbf{k+1}}$$

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{\mathbf{n+1}}{\mathbf{k+1}}$$

Application. Find the sum

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n-2)(n-1)n$$

Solution: Observe that $\binom{m}{3} = \frac{m!}{3!(m-3)!} = \frac{(m-2)\cdot(m-1)\cdot m}{6}$.
Thus

$$\frac{1 \cdot 2 \cdot 3}{6} + \frac{2 \cdot 3 \cdot 4}{6} + \cdots + \frac{(n-2)(n-1)n}{6}$$

$$= \binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3} = \binom{n+1}{3+1} = \binom{\mathbf{n+1}}{\mathbf{4}}$$

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n-2)(n-1)n = 6 \cdot \binom{n+1}{4} = \frac{(n-2)(n-1)n(n+1)}{4}$$