1. The density function of $X$ is given by

$$f(x) = \begin{cases} \frac{c}{x^4} & x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that $c = 3$.
(ii) Compute $E[X]$ and $Var(X)$.

2. The standard normal random variable $Z$ is characterized by its density function

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$ 

Compute $E[e^Z]$.

3. We know that for any finite collection of random variables (not necessarily independent), the expectation of their sum is equal to the sum of their expectations. If $X$ and $Y$ are independent random variables, prove

$$Var(X + Y) = Var(X) + Var(Y)$$

Hint: Recall that if $X$ and $Y$ are independent random variables, then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

4. Let $X_1$, $X_2$, and $X_3$ each be uniform random variables over $[0, 1]$. If in addition $X_1$, $X_2$, and $X_3$ are independent, find the probability density function $f_{X_1+X_2+X_3}$ of their sum $X_1 + X_2 + X_3$.

Hint: The density function $f_{X_1+X_2}$ of $X_1 + X_2$ was found in class:

$$f_{X_1+X_2}(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Next, notice that $f_{X_1+X_2+X_3}(a) = \int_{-\infty}^{\infty} f_{X_1+X_2}(x)f_{X_3}(a-x) \, dx$
5. An experiment consists of 1,210 independent Bernoulli trials with probability $p = \frac{1}{11}$ of success. Use the Central Limit Theorem for Bernoulli Trials of section 9.1 (the DeMoivre-Laplace theorem) and the table of values for the standard normal distribution to estimate the probability of the event that

$$\{98 \leq \text{the number of successes} \leq 116\}.$$  

Remember: it is best to consider $P\{97.5 \leq \text{the number of successes} \leq 116.5\}$.

6. As part of the gambling game at the “Magic Casino”, a fair coin ($p = \frac{1}{2}$) is flipped 90,000 times. Use the Central Limit Theorem for Bernoulli Trials of section 9.1 (the DeMoivre-Laplace theorem) and the table of values for the standard normal distribution to estimate the probability of the event that

$$\{45,032 \leq \text{the number of heads} \leq 45,069\}.$$ 

7. A fair die is rolled 18,000 times. Estimate the probability that ‘6’ comes up at least 3,060 times.