Topics:

- Permutations
- Combinations
- Generalized combinations.
- Binomial theorem.
- Multinomial theorem.
- Examples.
COMBINATORICS
Counting: Multiplicative rule
Two experiments are being performed.

- **Experiment 1:** Tossing a coin. Possible outcomes are H (heads) and T (tails). The number of outcomes \(n_1 = 2\).

- **Experiment 2:** Rolling a die. Possible outcomes are 1, 2, 3, 4, 5 and 6. The number of outcomes \(n_2 = 6\).

The total number of possible outcomes is

\[ n_1 \times n_2 = 12 \]

They are

\( (H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6) \)
Counting: Multiplicative rule

Similarly, if $k$ experiments are being performed.

- Experiment 1: $n_1$ possible outcomes
- Experiment 2: $n_2$ possible outcomes (per each outcome from experiment 1)
- Experiment 3: $n_3$ possible outcomes (per each outcome from experiments 1 and 2)
  
  : 

- Experiment $k$: $n_k$ possible outcomes (per each outcome from experiments 1 through $k-1$)

Then the total number of possible outcomes is

$$n_1 \times n_2 \times \cdots \times n_k$$
Permutations

Question: In how many ways can $n$ distinct objects be ordered?

Examples:
• The order in which $n$ runners finish a race.
• The number of five letter strings ($n = 5$) one can create with five letters, A, B, C, D and E, if each letter is used exactly once (BADCE is an example of such string).
The number of distinct strings one can create with \( n \) distinct symbols, \( A_1, A_2, \ldots, A_n \), if each symbol is used exactly once.

The number of ways a deck of \( n \) distinct cards can be ordered (card shuffling).

The number of ways \( n \) distinct people can be ordered in a line.

Each ordering of \( n \) distinct objects is called a **permutation**. We want to find the total number of possible permutations of \( n \) distinct objects.
**Permutations:** Counting all \( n \)-permutations can be done via \( n \) experiments.

**Example.** *Count the number of five letter strings one can create with five letters, A, B, C, D and E, if each letter is used exactly once.*

We have \( n = 5 \) spots to fill: _ _ _ _ __

- Experiment 1: Placing A. There are five available spots, hence there are \( n_1 = 5 \) possible outcomes of the experiment.

After we place A there are still four vacant spots in the string:

_ _ _ A _
• **Experiment 2:** Placing B. There are four available spots, hence there are \( n_2 = 4 \) possible outcomes of the experiment.

After we place B there are still three vacant spots in the string:

\[
\begin{array}{ccc}
B & \_ & A & \_ \\
\end{array}
\]

• **Experiment 3:** Placing C. There are \( n_3 = 3 \) possible outcomes of the experiment.
After we place C there are still two vacant spots in the string:

\[
\begin{array}{ccc}
B & C & A & \_ \\
\end{array}
\]
• Experiment 4: Placing D. There are \( n_4 = 2 \) possible outcomes of the experiment. After we place D there are still one vacant spots in the string:

\[
\text{B } \_ \text{ C A D}
\]

• Experiment 5: Placing E. There are \( n_5 = 1 \) possible outcomes of the experiment. After we place E there are still no vacant spots in the string:

\[
\text{B E C A D}
\]

By the multiplicative rule,

\[
n_1 \times n_2 \times n_3 \times n_4 \times n_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120
\]

is the total number of possible outcomes.
Observe that each outcome is a distinct string, and that each permutation is a possible outcome. Thus we counted all permutations of 5 distinct objects:

\[ 5 \times 4 \times 3 \times 2 \times 1 = 120 \]
**Permutations:** Counting all permutations of \( n \) distinct objects can be done via \( n \) experiments. Therefore the multiplicative rule implies the following.

**Proposition.** The total number of permutations of \( n \) distinct objects is

\[
 n! = 1 \times 2 \times \cdots \times n
\]

called \( n \) factorial.

Here, \( 1! = 1, \quad 2! = 1 \times 2 = 2, \)
\( 3! = 1 \times 2 \times 3 = 6, \quad 4! = 1 \times 2 \times 3 \times 4 = 24, \)
\( 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120 \quad \text{and so on.} \)

Also we let \( 0! = 1. \)
Combinations.

**Question:** How many $n$-long strings is possible to create with $k$ A’s and $n - k$ B’s?

**Answer:**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Another way to denote $\binom{n}{k}$ is $C(n, k)$ (reads “$n$ choose $k$”).

Other alternative notations: $C^k_n$ and $nC_k$
Combinations.

Example: How many strings of length 8 is it possible to create with 5 $A$s and 3 $B$s (such as $BAAAAABBA$)?

Answer:

$$ \binom{8}{5} = \frac{8!}{5!3!} = 56 $$
Why $\frac{8!}{5!3!}$? Let’s index the 5 $A$s and 3 $B$s as follows

$$A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3$$

So with the 8 different characters we can make $8!$ different strings such as $B_2A_3A_5A_2A_1B_1B_3A_4$

Alternatively, we can count as follows:

1. Arranging 5 $A$s and 3 $B$s into an eight letter string such as $BAAAAABBA$. There are $\binom{8}{5}$ ways.

2. Ordering five $A$s as in $BA_3A_5A_2A_1BBA_4$. This can be done in $5!$ different ways.

3. Ordering three $B$s as in $B_2A_3A_5A_2A_1B_1B_3A_4$. This can be done in $3!$ different ways.

Therefore, $8! = \binom{8}{5}5!3!$ and $\binom{8}{5} = \frac{8!}{5!3!}$
Why \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)? Let’s index \( k \) letters \( A \) and \( n-k \) letters \( B \) as follows

\[
A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_{n-k}
\]

With this \( n \) different characters we can make \( n! \) different permutations.

Alternatively, we can count as follows:

1. Arranging \( k \) letters \( A \) and \( n-k \) letters \( B \) into an \( n \) letter string can be done in \( \binom{n}{k} \) different ways.

2. Ordering \( k \) letters \( A \) can be done in \( k! \) different ways.

3. Ordering \( n-k \) letters \( B \) can be done in \( (n-k)! \) different ways.

Hence, \( n! = \binom{n}{k} k! (n-k)! \), and therefore, \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)
Properties of $\binom{n}{k}$.

- $\binom{n}{k} = \binom{n}{n-k}$

- $\binom{n}{0} = \binom{n}{n} = 1$

- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $0 < k < n$
Pascal’s triangle. 

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 \\
\binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 \\
\binom{2}{1} &= ? \\
\binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 \\
\binom{3}{1} &= ? \\
\binom{3}{2} &= ? \\
\binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 \\
\binom{4}{1} &= ? \\
\binom{4}{2} &= ? \\
\binom{4}{3} &= ? \\
\binom{4}{4} &= 1
\end{align*}
\]

...
Pascal’s triangle.

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 & \binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 & \binom{2}{1} &= 2 & \binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 & \binom{3}{1} &= ? & \binom{3}{2} &= ? & \binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 & \binom{4}{1} &= ? & \binom{4}{2} &= ? & \binom{4}{3} &= ? & \binom{4}{4} &= 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{align*}
\]
Pascal’s triangle.

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 \quad \binom{1}{1} = 1 \\
\binom{2}{0} &= 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1 \\
\binom{3}{0} &= 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = ? \quad \binom{3}{3} = 1 \\
\binom{4}{0} &= 1 \quad \binom{4}{1} = ? \quad \binom{4}{2} = ? \quad \binom{4}{3} = ? \quad \binom{4}{4} = 1 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]
**Pascal’s triangle.**

\[
{n \choose k} = {n-1 \choose k} + {n-1 \choose k-1}
\]

\[
{n \choose 0} = 1
\]

\[
{n \choose 1} = 1
\]

\[
{n \choose 2} = n
\]

\[
{n \choose 3} = \frac{n(n-1)(n-2)}{3!}
\]

\[
{n \choose 4} = \frac{n(n-1)(n-2)(n-3)}{4!}
\]

\[
{n \choose 5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}
\]

\[
{n \choose 6} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}
\]

...
Pascal’s triangle.

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1, \quad \binom{1}{1} = 1 \\
\binom{2}{0} &= 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1 \\
\binom{3}{0} &= 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1 \\
\binom{4}{0} &= 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = ?, \quad \binom{4}{3} = ?, \quad \binom{4}{4} = 1 \\
\vdots &\quad \vdots &\quad \vdots &\quad \vdots &\quad \vdots
\end{align*}
\]
Pascal's triangle.

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 & \binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 & \binom{2}{1} &= 2 & \binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 & \binom{3}{1} &= 3 & \binom{3}{2} &= 3 & \binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 & \binom{4}{1} &= 4 & \binom{4}{2} &= 6 & \binom{4}{3} &= ? & \binom{4}{4} &= 1 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]
Pascal’s triangle. 

\[
{n \choose k} = \left( {n-1 \choose k} \right) + \left( {n-1 \choose k-1} \right)
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1, \quad \binom{1}{1} = 1 \\
\binom{2}{0} &= 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1 \\
\binom{3}{0} &= 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1 \\
\binom{4}{0} &= 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots 
\end{align*}
\]
Generalized combinations.

**Question:** How many $n$-long strings is possible to create with $k_1$ A’s, $k_2$ B’s and $n - k_1 - k_2$ C’s?

**Answer:**

$$\binom{n}{k_1 + k_2} \cdot \binom{k_1 + k_2}{k_1} = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!}$$
Proof: There are two experiments.
(i) Placing $k = k_1 + k_2$ Œs and $n - k$ Cs in the $n$-long string:

$$\text{Œ Œ C Œ C C C Œ Œ C C}$$

This can be done in $\binom{n}{k}$ ways.

(ii) Replacing Œs with $k_1$ As and $k_2$ Bs:

$$\text{A B C A C C C A B C C}$$

This can be done in $\binom{k}{k_1}$ ways.

Answer:

$$\binom{n}{k} \cdot \binom{k}{k_1} = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!}$$
Generalized combinations.

Example: How many 24-long strings is possible to create with 7 A’s, 11 B’s and 6 C’s?

(e.g. BCBBBCABAAACCCBBBAABBBBBACA)

Answer:

\[
\binom{24}{18} \cdot \binom{18}{7} = \frac{24!}{7! \cdot 11! \cdot 6!}
\]
Generalized combinations.

Question: Count the number of different ways in which \( n \) objects of \( r \) different types can be ordered if there are

\[
\begin{align*}
  k_1 & \text{ objects of type 1,} \\
  k_2 & \text{ objects of type 2,} \\
  \vdots & \\
  k_r & \text{ objects of type } r.
\end{align*}
\]

All in all the total of \( k_1 + k_2 + \cdots + k_r = n \) objects.

Answer:

\[
\frac{n!}{k_1! \cdot k_2! \cdots \cdots k_r!}
\]
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

- Example.

\[(x + y)^2 = \binom{2}{0}x^0y^2 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^2y^0\]

\[= y^2 + 2xy + x^2\]
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

- **Example.**

\[(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0\]

\[= y^3 + 3xy^2 + 3x^2y + x^3\]
**Binomial Theorem:** 

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Proof.** The Binomial theorem follows easily from the definition of $\binom{n}{k}$.

Take $n = 2$,

$$(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy$$

Observe that we sum up over all possible 2-long strings that can be created with $x$ and $y$,

$$xx, xy, yx, \text{ and } yy$$

- There are $\binom{2}{2} = 1$ strings with two $x$ and no $y$
- There are $\binom{2}{1} = 2$ strings with one $x$ and one $y$
- There are $\binom{2}{0} = 1$ strings with no $x$ and two $y$

We consolidate the terms to obtaining

$$(x + y)^2 = x^2 + 2xy + y^2$$
Take \( n = 3 \),

\[
(x + y)^3 = (x + y)(x + y)(x + y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy
\]

Observe that we sum up over all possible 3-long strings that can be created with \( x \) and \( y \).

- There are \( \binom{3}{3} = 1 \) strings with three \( x \) and no \( y \)
- There are \( \binom{3}{2} = 3 \) strings with two \( x \) and one \( y \)
- There are \( \binom{3}{1} = 3 \) strings with one \( x \) and two \( y \)
- There are \( \binom{3}{0} = 1 \) strings with no \( x \) and three \( y \)

We consolidate the terms to obtaining

\[
(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3
\]
When we factor \((x + y)^n\), each \((x + y)\) contributes either \(x\) or \(y\) to the resulting \(n\)-long string:

\[
(x + y)(x + y)(x + y)\ldots(x + y)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \ldots \quad \downarrow
\]

\[
x \text{ or } y \quad x \text{ or } y \quad x \text{ or } y \quad \ldots \quad x \text{ or } y
\]

Multiplying out \((x + y)(x + y)(x + y)\ldots(x + y)\ldots(x + y)\) we end up with all possible distinct \(n\)-long strings made of \(x\) and \(y\).

We consolidate the terms that have the same number of \(x\) and \(y\):

- There are \(\binom{n}{k}\) strings with \(k\) letters \(x\) and \(n - k\) letters \(y\).

Hence

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]
Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

This is why the numbers $\binom{n}{k}$ are also known as **binomial coefficients**.

- **Example.** Find the coefficient in front of $x^7 y^9$ in the expansion of $(x + y)^{16}$.

- **Answer:** $\binom{16}{7} = 11,440$
Binomial Theorem: \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

- **Example.** Find \(\sum_{k=0}^{n} \binom{n}{k}\).

- **Solution:** Observe that if we use the Binomial Theorem with \(x = y = 1\), the left hand side \(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\) becomes \(\sum_{k=0}^{n} \binom{n}{k}\).

  Thus

  \[
  \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1 + 1)^n = 2^n
  \]
Generalized combinations.

Question: Count the number of different ways in which \( n \) objects of \( r \) different types can be ordered if there are

\[
\begin{align*}
  k_1 & \text{ objects of type 1,} \\
k_2 & \text{ objects of type 2,} \\
\vdots & \\
k_r & \text{ objects of type } r.
\end{align*}
\]

All in all the total of \( k_1 + k_2 + \cdots + k_r = n \) objects.

Answer:

\[
\frac{n!}{k_1! \cdot k_2! \cdots k_r!}
\]
Multinomial Theorem

\[(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1 \geq 0, \ldots, k_r \geq 0 \atop k_1 + k_2 + \cdots + k_r = n} \frac{n!}{k_1! \cdot k_2! \cdots k_r!} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}\]

Example:

\[(x + y + z)^2 = \frac{2!}{2!0!0!} x^2 y^0 z^0 + \frac{2!}{0!2!0!} x^0 y^2 z^0 + \frac{2!}{0!0!2!} x^0 y^0 z^2\]

\[+ \frac{2!}{0!1!1!} x^0 y^1 z^1 + \frac{2!}{1!0!1!} x^1 y^0 z^1 + \frac{2!}{1!1!0!} x^1 y^1 z^0\]

\[= x^2 + y^2 + z^2 + 2yz + 2xz + 2xy\]

Multinomial Theorem can be proved in the same way as the Binomial Theorem. Alternatively, it can be proved using the Binomial Theorem.
**Question:** How many strings can be formed with the following letters?

\[ S \ H \ E \ S \ E \ L \ L \ S \ S \ E \ A \ S \ H \ E \ L \ L \ S \]

**Solution:** There are

\[ k_1 = 1 \text{ letter A}, \]
\[ k_2 = 4 \text{ letters E}, \]
\[ k_3 = 2 \text{ letters H}, \]
\[ k_4 = 4 \text{ letters L}, \]
\[ k_5 = 6 \text{ letters S} \]

Therefore there must be

\[
\frac{17!}{1!4!2!4!6!}
\]

different strings made with these letters.
**Question:** In how many ways can 52 different cards be dealt between four players (call them Player 1, Player 2, Player 3, and Player 4) where each gets exactly 13 cards?

**Solution:** Line up the cards: $[1, 2, \ldots, 52]$.

Each card is assigned a player. Thus there will be 4 types of cards: the ones that belong to Player 1, the ones that belong to Player 2, the ones that belong to Player 3, and the ones that belong to Player 4. Hence there are $\frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}$ ways to deal the deck.