MTH 361

Lectures 20 - 21

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Topics:

- Moment generating functions.
- Review.

Moment generating functions.

Definition. For a given random variable X, the function

$$M_X(s) = E\left[e^{sX}\right]$$

is called the moment generating function (m.g.f.).

Properties: • $M_X(0) = 1$.

• $$M_X(s) = E\left[e^{sX}\right] = \begin{cases} \sum\limits_{x:\, p_X(x)>0} e^{sx} p_X(x) & \text{if } X \text{ is discrete,} \\ \int\limits_{-\infty}^{\infty} e^{sx} f_X(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

• The derivatives of $M_X(s)$ are computed as follows

$$M'_X(s) = \frac{d}{ds} E[e^{sX}] = E[X e^{sX}] \quad \text{and}$$

$$M_X^{(n)}(s) = \frac{d^n}{ds^n} E[e^{sX}] = E\left[\frac{d^n}{ds^n} e^{sX}\right] = E[X^n e^{sX}].$$

Thus, $M_X^{(n)}(0) = E[X^n]$ (the n^{th} moment), and

$$E[X] = M'_X(0), \quad E[X^2] = M''_X(0), \quad Var(X) = M''_X(0) - \left(M'_X(0)\right)^2.$$

Moment generating functions.

Definition. For a given random variable X, the function

$$M_X(s) = E\left[e^{sX}\right]$$

is called the moment generating function (m.g.f.).

An important property of $M_X(s)$: If X and Y are independent random variables with the respective moment generating functions $M_X(s)$ and $M_Y(s)$, then the moment generating function of $X + Y$ is

$$M_{X+Y}(s) = E\left[e^{s(X+Y)}\right] = E\left[e^{sX}e^{sY}\right] = E\left[e^{sX}\right]E\left[e^{sY}\right] = M_X(s)\,M_Y(s).$$

Hence, if $X_1, X_2, \ldots, X_n$ are independent random variables, then the moment generating function of $X = X_1 + X_2 + \ldots + X_n$ equals

$$M_X(s) = M_{X_1}(s) \cdot M_{X_2}(s) \cdot \ldots \cdot M_{X_n}(s).$$

Moment generating functions.

Example. Consider a Bernoulli random variable X with parameter $p \in [0,1]$, i.e., $X \sim Bernoulli(p)$. Then,

$$M_X(s) = E[e^{sX}] = \sum_{k=0,1} e^{sk} p_X(k) = 1 \cdot (1-p) + e^s \cdot p.$$

Hence,

$$M_X(s) = 1 - p + pe^s \quad \text{with the domain} \quad s \in \mathbb{R}.$$

Example. Consider a binomial random variable X with parameters (n,p), i.e., $X \sim Binomial(n,p)$. Then,

$$X = X_1 + X_2 + \ldots + X_n,$$

where $X_1, X_2, \ldots, X_n$ are independent $Bernoulli(p)$ random variables. Thus,

$$M_X(s) = M_{X_1}(s) \cdot M_{X_2}(s) \cdot \ldots \cdot M_{X_n}(s) = \Big(1 - p + pe^s\Big)^n, \qquad s \in \mathbb{R}.$$

Hence, $E[X] = M'_X(0) = np$, $\quad E[X^2] = M''_X(0) = np + n(n-1)p^2$,

$$\text{and} \qquad Var(X) = E[X^2] - \big(E[X]\big)^2 = np(1-p).$$

Moment generating functions.

Example. Consider a binomial random variable X with parameters (n, p), i.e., $X \sim Binomial(n, p)$. Then,

$$M_X(s) = \left(1 - p + pe^s\right)^n, \qquad s \in \mathbb{R}.$$

Alternative derivation via Binomial Theorem:

$$M_X(s) = \sum_{k=0}^{n} e^{sk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left(pe^s\right)^k (1-p)^{n-k} = \left(1 - p + pe^s\right)^n$$

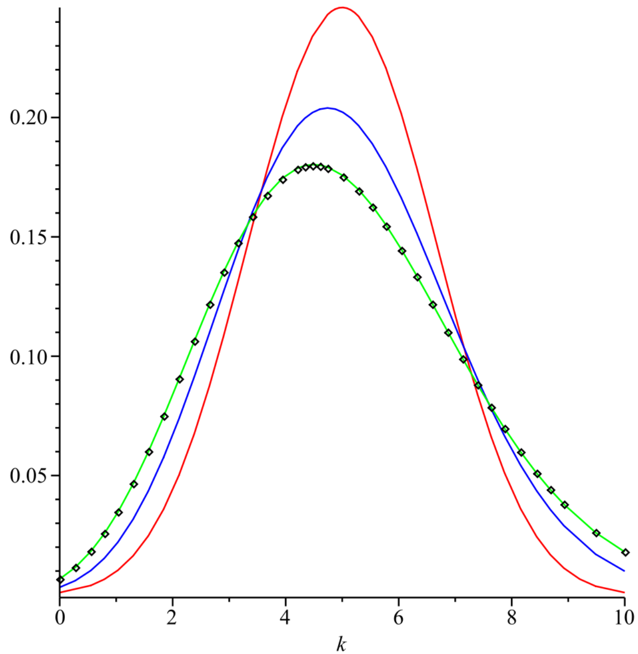
Example. Consider a Poisson random variable X with parameter $\lambda > 0$. i.e., $X \sim Poisson(\lambda)$. Then,

$$M_X(s) = E[e^{sX}] = \sum_{k=0}^{\infty} e^{sk} p_{\mathsf{x}}(k) = \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(\lambda e^s\right)^k}{k!} = e^{-\lambda} e^{\lambda e^s}.$$

Hence,

$$M_X(s) = \exp\left\{\lambda(e^s - 1)\right\}, \qquad s \in \mathbb{R}.$$

Poisson vs Binomial.



Picture credit: Wikipedia.org

Dots: Poisson($\lambda = 5$)

Red: Binomial($n = 10,\ p = \frac{1}{2}$)

Blue: Binomial($n = 20,\ p = \frac{1}{4}$)

Green: Binomial($n = 1000,\ p = \frac{1}{200}$)

Poisson vs Binomial.

Let $\lambda > 0$ be given. Suppose Y is a Poisson random variable with parameter λ and S_n is a Binomial random variable with parameters n and $p = \frac{\lambda}{n}$.

• **Theorem.** For a given integer $k \geq 0$, $\lim_{n\to\infty} P(S_n = k) = P(Y = k)$.

Thus, for n large enough, $P(S_n = k) \approx P(Y = k)$.

Alternative proof: $\forall s \in \mathbb{R}$,

$$M_{S_n}(s) = \left(1 - p + pe^s\right)^n = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}e^s\right)^n = \left(1 + \frac{\lambda(e^s - 1)}{n}\right)^n$$

Hence,

$$\lim_{n\to\infty} M_{S_n}(s) = \lim_{n\to\infty} \left(1 + \frac{\lambda(e^s - 1)}{n}\right)^n = e^{\lambda(e^s-1)} = M_Y(s).$$

Theorem. The cumulative distribution function $F_X(x)$ is unique for a m.g.f. $M_X(s)$. Moreover, if $\lim_{n\to\infty} M_{X_n}(s) = M_X(s)$, then the cumulative distribution functions also converge, i.e.,

$$\lim_{n\to\infty} F_{X_n}(a) = F_X(a) \qquad \forall a \in \mathbb{R}$$

Moment generating functions.

Example. Consider a standard normal random variable Z, i.e., $Z \sim \mathcal{N}(0,1)$. Then, its moment generating function equals

$$M_Z(s) = E[e^{sZ}] = \int_{-\infty}^{\infty} e^{sx} f(x)\ dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{1}{2}x^2}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2sx)}\ dx = e^{\frac{1}{2}s^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s)^2}\ dx$$

Hence,

$$M_Z(s) = \exp\left\{\frac{s^2}{2}\right\}, \qquad s \in \mathbb{R}.$$

Theorem. The cumulative distribution function $F_X(x)$ is unique for a m.g.f. $M_X(s)$. Moreover, if $\lim_{n\to\infty} M_{X_n}(s) = M_X(s)$, then the cumulative distribution functions also converge, i.e.,

$$\lim_{n\to\infty} F_{X_n}(a) = F_X(a) \qquad \forall a \in \mathbb{R}$$

Moment generating functions.

Example. Consider a geometric random variable X with parameter $p \in (0,1)$, i.e., $X \sim Geometric(p)$. Then,

$$M_X(s) = E[e^{sX}] = \sum_{k=1}^{\infty} e^{sk} p_X(k) = \sum_{k=1}^{\infty} e^{sk}(1-p)^{k-1}p$$

$$= pe^s \sum_{k=1}^{\infty} \Big((1-p)e^s\Big)^{k-1} = \frac{pe^s}{1-(1-p)e^s} \quad \text{when} \quad (1-p)e^s < 1.$$

Hence,

$$M_X(s) = \frac{pe^s}{1-(1-p)e^s}, \qquad s \in \Big(-\infty, -\ln(1-p)\Big).$$

Differentiating $M_X(s) = \frac{pe^s}{1-(1-p)e^s}$ we obtain

$$M'_X(s) = \frac{pe^s}{\Big(1-(1-p)e^s\Big)^2}, \qquad M''_X(s) = \frac{pe^s + p(1-p)e^{2s}}{\Big(1-(1-p)e^s\Big)^3}.$$

Therefore, $E[X] = M'_X(0) = \frac{1}{p}$, $\quad E[X^2] = M''_X(0) = \frac{2-p}{p^2}$, and

$$Var(X) = E[X^2] - \Big(E[X]\Big)^2 = \frac{1-p}{p^2}.$$

Moment generating function for $X \sim Exponential(\lambda)$

Example. Consider a exponential random variable X with parameter $\lambda > 0$, i.e., $X \sim Exponential(\lambda)$. Then, for $s < \lambda$,

$$M_X(s) = \int_0^\infty e^{sx} \lambda e^{-\lambda x}\, dx = \frac{\lambda}{\lambda - s} \int_0^\infty (\lambda - s) e^{-(\lambda - s)x}\, dx$$

Let $y = (\lambda - s)x$, then

$$M_X(s) = \frac{\lambda}{\lambda - s} \int_0^\infty e^{-y}\, dy = \frac{\lambda}{\lambda - s}, \qquad s \in (-\infty, \lambda).$$

Here,

$$M_X^{(n)}(s) = \frac{n!\lambda}{(\lambda - s)^{n+1}} \quad \text{implies} \quad E[X^n] = M_X^{(n)}(0) = \frac{n!}{\lambda^n},$$

and therefore, $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

Central Limit Theorem.

• **Central Limit Theorem (CLT).** Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean μ and variance σ^2. Then,

$$\lim_{n\to\infty} P(a \le Y_n \le b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \quad \Leftrightarrow \quad \lim_{n\to\infty} F_{Y_n}(a) = \Phi(a),$$

where $Y_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sqrt{n}\sigma}$ and $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$ is the standard normal cumulative distribution function.

The de Moivre-Laplace Theorem is a case of CLT when $X_1, X_2, \ldots$ are independent Bernoulli random variables with the same parameter $p \in (0,1)$.

• **de Moivre-Laplace Theorem.** Let S_n be a binomial random variable with parameters (n,p), then

$$\lim_{n\to\infty} F_{Y_n}(a) = \Phi(a), \qquad \text{where} \quad Y_n = \frac{S_n - np}{\sqrt{np(1-p)}}.$$

Thus, it is sufficient to show that

$$\lim_{n\to\infty} M_{Y_n}(s) = \exp\left\{\frac{s^2}{2}\right\} \qquad \text{- m.g.f. for } \mathcal{N}(0,1).$$

de Moiver-Laplace Theorem via m.g.f.

Proof. Consider $S_n \sim Binomial(n,p)$ and let $Y_n = \frac{S_n - np}{\sqrt{np(1-p)}}$.

Then, $\quad E[Y_n] = 0 \quad$ and $\quad Var(Y_n) = 1$.

The moment generating function

$$M_{Y_n}(s) = \exp\left\{-s\frac{np}{\sqrt{np(1-p)}}\right\} \cdot M_{S_n}\left(\frac{s}{\sqrt{np(1-p)}}\right)$$

$$= \exp\left\{-s\frac{np}{\sqrt{np(1-p)}}\right\} \cdot \left(1 - p\left[1 - \exp\left\{\frac{s}{\sqrt{np(1-p)}}\right\}\right]\right)^n$$

and

$$\ln M_{Y_n}(s) = -s\frac{np}{\sqrt{np(1-p)}} + n\ln\left(1 - p\left[1 - \exp\left\{\frac{s}{\sqrt{np(1-p)}}\right\}\right]\right)$$

de Moiver-Laplace Theorem via m.g.f.

Consider $S_n \sim Binomial(n,p)$ and let $Y_n = \frac{S_n - np}{\sqrt{np(1-p)}}$.

$$\ln M_{Y_n}(s) = -s\frac{np}{\sqrt{np(1-p)}} + n\ln\left(1 - p\left[1 - \exp\left\{\frac{s}{\sqrt{np(1-p)}}\right\}\right]\right),$$

where

$$\alpha := 1 - \exp\left\{\frac{s}{\sqrt{np(1-p)}}\right\} = -\frac{s}{\sqrt{np(1-p)}} - \frac{s^2}{2np(1-p)} + O\left(\frac{1}{n^{3/2}}\right)$$

and therefore,

$$\ln(1-p\alpha) = -p\alpha - \frac{p^2\alpha^2}{2} + O\left(\frac{1}{n^{3/2}}\right) = \frac{ps}{\sqrt{np(1-p)}} + \frac{s^2}{2n(1-p)} - \frac{ps^2}{2n(1-p)} + O\left(\frac{1}{n^{3/2}}\right)$$

$$= \frac{ps}{\sqrt{np(1-p)}} + \frac{s^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)$$

de Moiver-Laplace Theorem via m.g.f.

Consider $S_n \sim Binomial(n, p)$ and let $Y_n = \frac{S_n - np}{\sqrt{np(1-p)}}$.

$$\ln M_{Y_n}(s) = -s\frac{np}{\sqrt{np(1-p)}} + n\ln(1-p\alpha),$$

where

$$\alpha := 1-\exp\left\{\frac{s}{\sqrt{np(1-p)}}\right\} = -\frac{s}{\sqrt{np(1-p)}} - \frac{s^2}{2np(1-p)} + O\left(\frac{1}{n^{3/2}}\right)$$

and

$$\ln(1-p\alpha) = \frac{ps}{\sqrt{np(1-p)}} + \frac{s^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)$$

Thus,

$$\ln M_{Y_n}(s) = \frac{s^2}{2} + O\left(\frac{1}{n^{1/2}}\right)$$

and

$$\lim_{n\to\infty} M_{Y_n}(s) = \exp\left\{\frac{s^2}{2}\right\} \quad \text{- m.g.f. for } \mathcal{N}(0,1).$$

Hence, $\lim_{n\to\infty} F_{Y_n}(a) = \Phi(a)$.

Review.

Problem 5 on p.71. Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(x) = \lambda e^{-\lambda x} \qquad (x \geq 0),$$

where $\lambda = 1$, so that the probability $P(0 \leq X \leq T)$ that a particle will appear in the next T seconds is $P(0 \leq X \leq T) = \int_0^T \lambda e^{-\lambda x} dx$

Find the probability that a particle will appear

(a) within the next second.

(b) within the next 3 seconds.

(c) between 3 and 4 seconds from now.

(d) after 4 seconds from now.

Review.

Problem 1 on p.278. Let X be a random variable with range $[-1, 1]$ and let $f_{\mathrm{x}}(x)$ be the density function of X. Find $E[X]$ and $Var(X)$ if, for $|x| < 1$,

(a) $f_{\mathrm{x}}(x) = \frac{1}{2}$

(b) $f_{\mathrm{x}}(x) = |x|$

(c) $f_{\mathrm{x}}(x) = 1 - |x|$

(d) $f_{\mathrm{x}}(x) = \frac{3}{2}x^2$

Review.

Problem 3 on p.278. The lifetime, measured in hours, of the ACME super light bulb is a random variable T with density function $f(x) = \lambda^2 x e^{-\lambda x}$, where $\lambda = 0.05$. What is the expected lifetime of this light bulb? What is its variance?

Review.

Problem 4 on p.278. Let X be a random variable with range $[-1,1]$ and density function $f_X(x) = ax + b$ if $|x| < 1$,

(a) Show that if $\int_{-1}^{1} f_X(x)dx = 1$, then $b = \frac{1}{2}$

(b) Show that if $f_X(x) \geq 0$, then $-\frac{1}{2} \leq a \leq \frac{1}{2}$.

(c) Show that $E[X] = \frac{2}{3}a$, and hence that $-\frac{1}{3} \leq E[X] \leq \frac{1}{3}$.

(d) Show that $Var(X) = \frac{2}{3}b - \frac{4}{9}a^2 = \frac{1}{3} - \frac{4}{9}a^2$.

Review.

Problem. Let X and Y be two independent random variables, each exponential with the same parameter $\lambda > 0$. Show that their sum, $X+Y$ is distributed via the following density function

$$f_{\mathrm{x+y}}(x) = \lambda^2 x e^{-\lambda x} \qquad (x \geq 0)$$

Review.

Problem 2 on p.219. Choose a number U from the interval $[0,1]$ with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = \frac{1}{U+1}$

(b) $Y = \log(U+1)$

Review.

Problem 10 on p.220. Let U, V be random numbers chosen independently from the interval $[0, 1]$. Find the cumulative distribution and density for the random variables

(a) $Y = \max(U, V)$

(b) $Y = \min(U, V)$

Review.

Problem 16 on p.221. Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function F_X for X?

(c) What is the probability that $X < \frac{1}{4}$?