

MTH 361

Lectures 14 - 16

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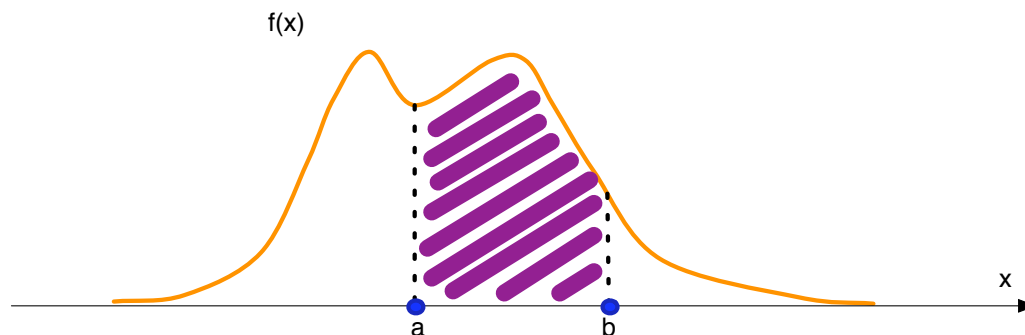
Topics:

- Continuous random variables.
- Exponential random variables.
- Uniform random variable.
- Normal random variable.
- Expectation of continuous variables.
- Variance of continuous variables.

Continuous random variables.

Definition. We say that X is a **continuous random variable** if there exists a nonnegative function $f(x)$ defined for all real x such that for any $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$



Such function $f(x)$ is the **probability density function** of X .

Continuous random variables.

Definition. We say that X is a **continuous random variable** if there exists a nonnegative function $f(x)$ defined for all real x such that for any $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Properties:

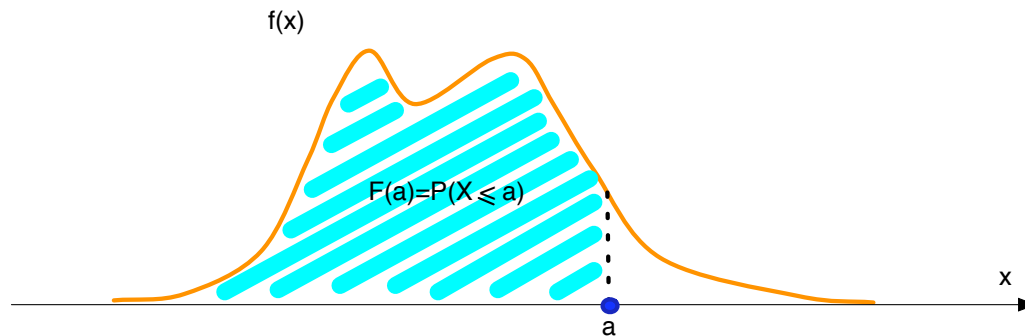
- $\int_{-\infty}^{\infty} f(x)dx = P(-\infty < X < \infty) = 1$, where $\int_{-\infty}^{\infty} f(x)dx = \lim_{a \rightarrow \infty} \int_{-a}^a f(x)dx$
- $P(X = a) = \int_a^a f(x)dx = 0$ for any real a .
- Hence

$$P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b) = \int_a^b f(x)dx$$

Continuous random variables.

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **cumulative distribution function** (cdf) is

$$F(a) = P(X \leq a) = \int_{-\infty}^a f(x) dx$$



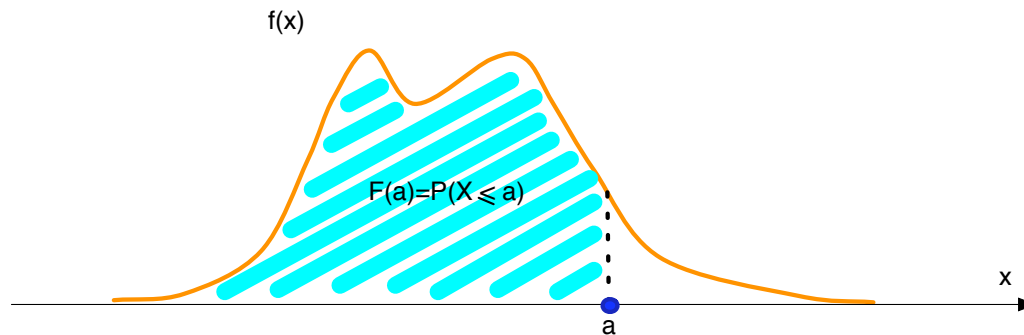
- Here, for any $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x) dx = \int_{-\infty}^b f(x) dx - \int_{-\infty}^a f(x) dx = F(b) - F(a)$$

Continuous random variables.

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **cummulative distribution function** (cdf) is

$$F(a) = P(X \leq a) = \int_{-\infty}^a f(x) dx$$



- Here, for any real a , the derivative

$$F'(a) = \frac{d}{da} \int_{-\infty}^a f(x) dx = f(a)$$

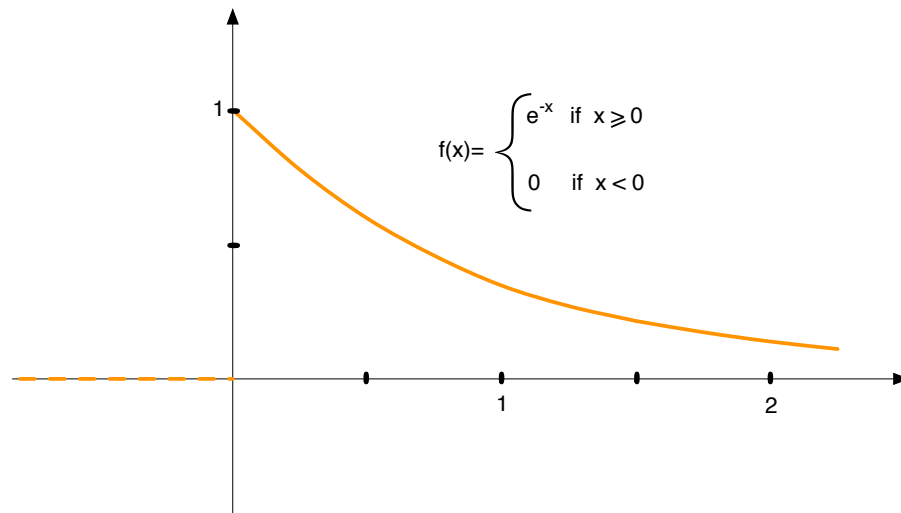
vis the Fundamental Theorem of Calculus.

Continuous random variables.

• **Example.** Let X be a continuous random variable with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.



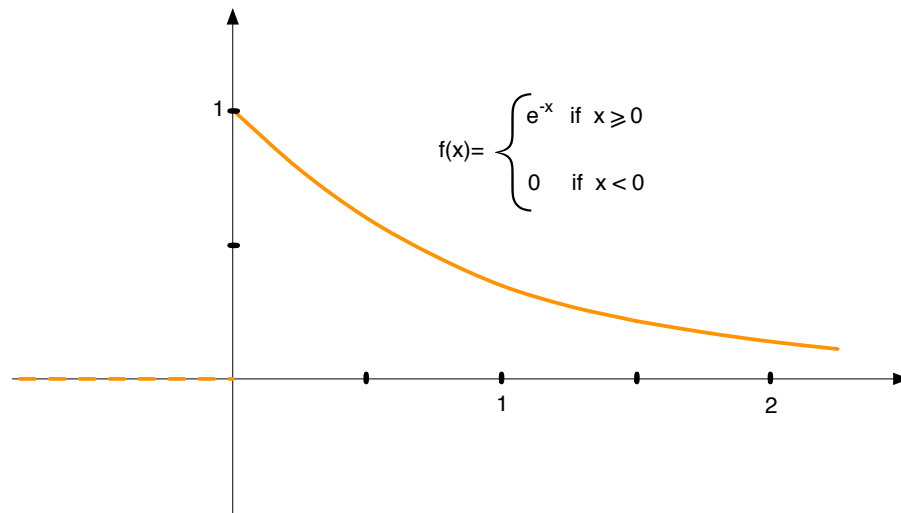
Continuous random variables.

- **Example (continued).** We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.

Solution: Here $f(x)$ is nonnegative and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^0 0 \cdot dx + \int_0^{\infty} e^{-x} dx = 0 + \left[-e^{-x} \right]_0^{\infty} = 1$$

Thus $f(x)$ is indeed a probability density function.

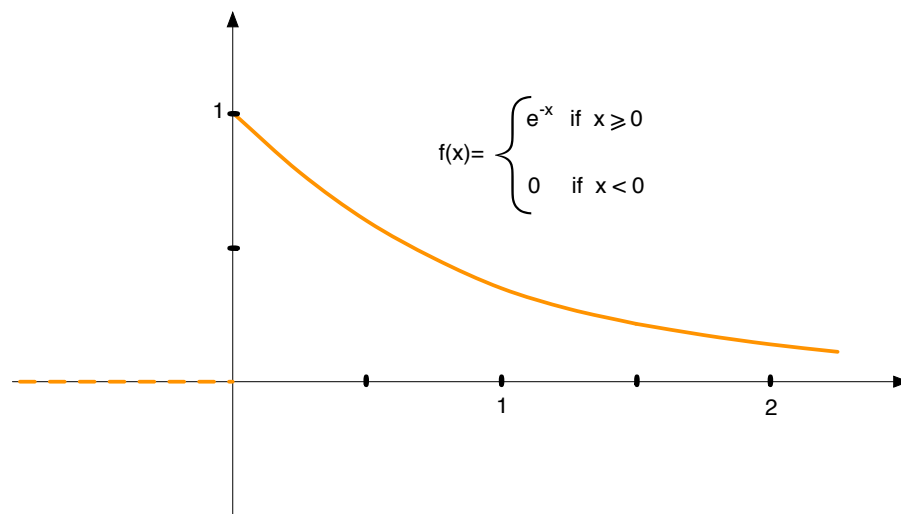


Continuous random variables.

- **Example (continued).** We want to check $f(x)$ is indeed a probability density function, and find $P(1 < X < 2)$.

Solution:

$$P(1 < X < 2) = \int_1^2 f(x) dx = \int_1^2 e^{-x} dx = \left[-e^{-x} \right]_1^2 = e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = 0.232544158 \dots$$

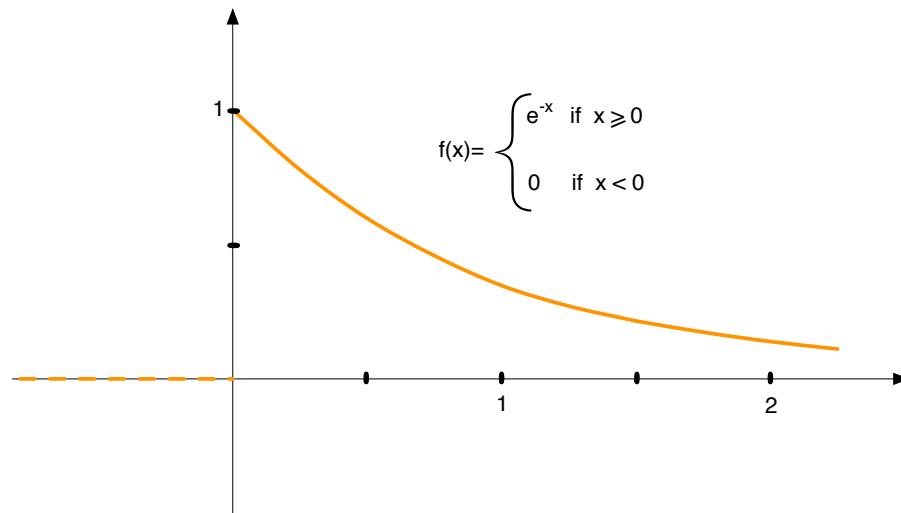


Exponential random variables.

- Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an **exponential random variable** with parameter λ .



Exponential random variables.

- Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an **exponential random variable** with parameter λ .

- Check:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 \cdot dx + \int_0^{\infty} \lambda e^{-\lambda x} dx = 0 + \left[-e^{-\lambda x} \right]_0^{\infty} = 1$$

Thus $f(x)$ is indeed a probability density function.

- Note: Exponential random variable is a continuous analogue to the geometric random variable. In particular it also satisfies the following *memoryless* property:

$$P(X > a + b \mid X > a) = P(X > b)$$

for any two positive a and b .

Uniform random variables.

- Consider an interval $[\alpha, \beta]$, where $\alpha < \beta$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Then X is said to be an **uniform random variable** over $[\alpha, \beta]$.

- Check:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{\alpha} 0 \cdot dx + \int_{\alpha}^{\beta} \frac{dx}{\beta - \alpha} + \int_{\beta}^{\infty} 0 \cdot dx = 0 + \left[\frac{x}{\beta - \alpha} \right]_{\alpha}^{\beta} + 0 \\ &= \frac{\beta}{\beta - \alpha} - \frac{\alpha}{\beta - \alpha} = 1 \end{aligned}$$

Thus $f(x)$ is indeed a probability density function.

Continuous random variables.

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **expectation** is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Properties:

- For any real valued function g , $g(X)$ will also be a **random variable**, and

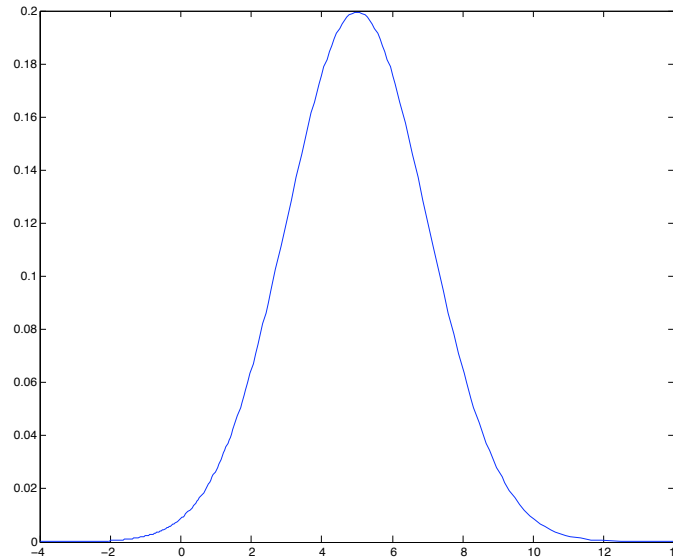
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **Markov inequality.** If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

- **Chebyshev inequality.** If X is a random variable with finite mean μ and variance, then for any $\kappa > 0$,

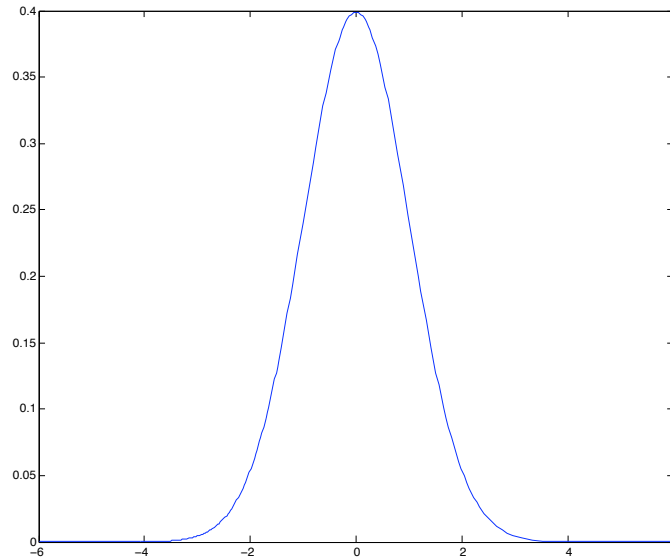
$$P(|X - \mu| \geq \kappa) \leq \frac{\text{Var}(X)}{\kappa^2}$$

Normal random variables.

X is a **Normal (Gaussian) random variable** with parameters μ and σ^2 if its density function is

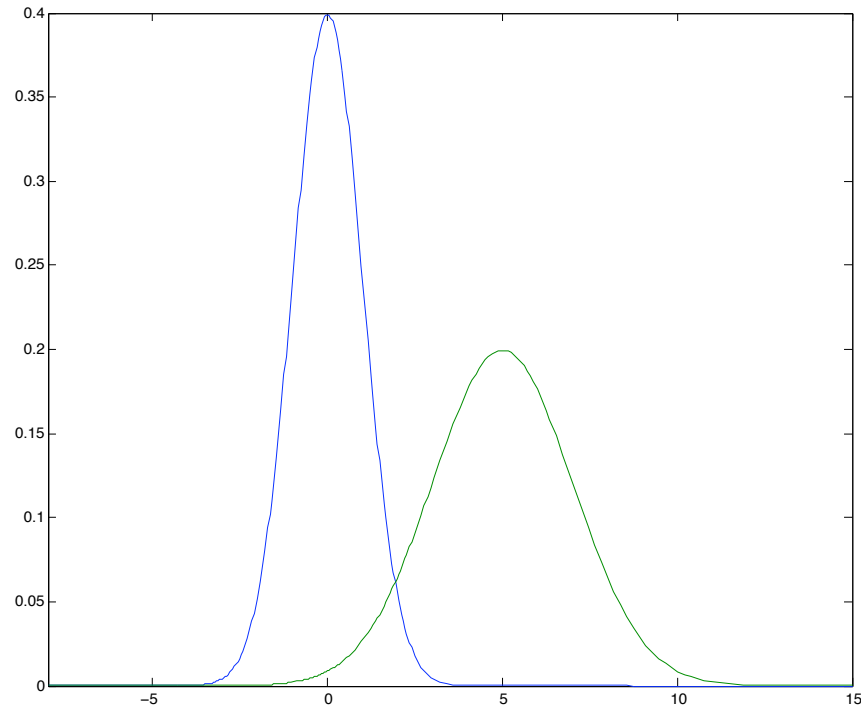
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

Normal random variables.



A normal random variable X is said to be **standard normal** if its parameters $\mu = 0$ and $\sigma^2 = 1$, and therefore

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

Normal random variables.

$\mathcal{N}(\mu, \sigma^2)$ denotes normal distribution with parameters μ and σ^2 . Here I plotted two normal densities, one standard normal $\mathcal{N}(0, 1)$ and one $\mathcal{N}(5, 4)$.

Normal random variables.

We need to check that the standard normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

integrates to 1.

Here let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. Then

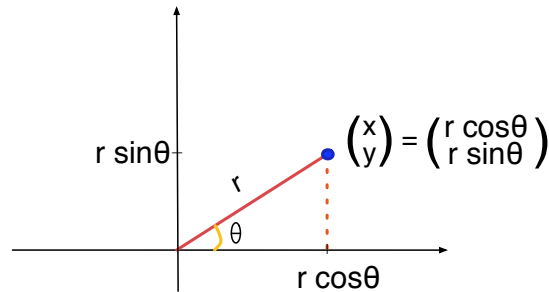
$$I^2 = \int_{-\infty}^{\infty} \left(e^{-\frac{y^2}{2}} \cdot I \right) dy = \int_{-\infty}^{\infty} \left(e^{-\frac{y^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot e^{-\frac{x^2}{2}} dx \right) dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot e^{-\frac{x^2}{2}} dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy$$

Normal random variables.

We let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$, and showed that $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy$

Use **polar coordinates**: let $x = r \cos \theta$ and $y = r \sin \theta$, where $0 \leq \theta < 2\pi$ and $0 \leq r < \infty$.



Here

$$dx dy = |J| dr d\theta = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| dr d\theta = \left| \det \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \right| dr d\theta = r dr d\theta$$

and

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = \int_0^{2\pi} \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty} d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Normal random variables.

So $I^2 = 2\pi$ and $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

Hence, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

Check: For a general $\mathcal{N}(\mu, \sigma^2)$ random variable,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1,$$

where we let $y = \frac{x-\mu}{\sigma}$.

Expectation and variance.

Recall

Definition. Let X be a continuous random variable with density function $f(x)$. Then its **expectation** is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Then

- For any real valued function g , $g(X)$ will also be a **random variable**, and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

and

- Given constants α and β ,

$$E[\alpha X + \beta] = \alpha E[X] + \beta$$

Expectation and variance.

Now, let X be a random variable with mean $E[X] = \mu$.

- **Definition.** The **variance** of X is

$$\text{Var}(X) = E[(X - \mu)^2]$$

- **Definition.** The **standard deviation** of X is

$$\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{E[(X - \mu)^2]}$$

Finally,

- **Theorem.** The **variance** of X is

$$\text{Var}(X) = E[X^2] - \mu^2$$

Exponential random variables.

Example. Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an **exponential random variable** with parameter λ .

We want to find its expectation.

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^{\infty} \lambda x \cdot e^{-\lambda x} dx$$

- Recall the **integration by parts formula**,

$$\int uv' dx = uv - \int u'v dx$$

Here we let $u(x) = x$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 1$ and $v'(x) = \lambda e^{-\lambda x}$, and
$$\int_0^{\infty} \lambda x \cdot e^{-\lambda x} dx = \int_0^{\infty} u(x)v'(x) dx$$

Exponential random variables.

Example (continued).

- Recall the **integration by parts formula**,

$$\int uv' dx = uv - \int u'v dx$$

Here we let $u(x) = x$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 1$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int_0^{\infty} \lambda x \cdot e^{-\lambda x} dx = \int_0^{\infty} u(x)v'(x) dx$

$$\begin{aligned} E[X] &= \int_0^{\infty} \lambda x \cdot e^{-\lambda x} dx = \left[-x e^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x}) dx = \int_0^{\infty} e^{-\lambda x} dx \\ &= \frac{1}{\lambda} \cdot \int_0^{\infty} \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \end{aligned}$$

as we have shown that $\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = 1$.

Normal random variables.

Example. Let X be a **Normal (Gaussian) random variable** with parameters μ and σ^2 . Its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

Then

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x - \mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x - \mu}{\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \end{aligned}$$

Normal random variables.

Example (continued). Let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \cdot e^{-\frac{y^2}{2}} dy + \mu = \mu$$

since $y \cdot e^{-\frac{y^2}{2}}$ is an **odd** function, implying

$$\int_{-\infty}^0 y \cdot e^{-\frac{y^2}{2}} dy = - \int_0^{\infty} y \cdot e^{-\frac{y^2}{2}} dy$$

Recall that $g(x)$ is an odd function if $g(-x) = -g(x)$.

Exponential random variables.

Example. Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an **exponential random variable** with parameter λ . We know its expectation $E[X] = \frac{1}{\lambda}$.

We want to find its variance, $Var(X) = E[X^2] - \left(E[X]\right)^2$

• Here

$$Var(X) = E[X^2] - \frac{1}{\lambda^2} = \int_0^{\infty} \lambda x^2 \cdot e^{-\lambda x} dx - \frac{1}{\lambda^2}$$

Once again, recall the **integration by parts formula**,

$$\int uv' dx = uv - \int u'v dx$$

Here we let $u(x) = x^2$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 2x$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int_0^{\infty} \lambda x^2 \cdot e^{-\lambda x} dx = \int_0^{\infty} u(x)v'(x)dx$

Exponential random variables.

Example (continued). Once again, recall the **integration by parts formula**,

$$\int uv' \, dx = uv - \int u'v \, dx$$

Here we let $u(x) = x^2$ and $v(x) = -e^{-\lambda x}$. Then $u'(x) = 2x$ and $v'(x) = \lambda e^{-\lambda x}$, and $\int_0^{\infty} \lambda x^2 \cdot e^{-\lambda x} dx = \int_0^{\infty} u(x)v'(x) dx$

$$\text{Var}(X) = \int_0^{\infty} \lambda x^2 \cdot e^{-\lambda x} dx - \frac{1}{\lambda^2} = \left[-x^2 e^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} (-2x e^{-\lambda x}) dx - \frac{1}{\lambda^2}$$

$$= 2 \int_0^{\infty} x e^{-\lambda x} dx - \frac{1}{\lambda^2} = \frac{2}{\lambda} \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2} = \frac{2}{\lambda} \cdot E[X] - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Normal random variables.

Example. Let X be a **Normal (Gaussian) random variable** with parameters μ and σ^2 . Its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

We have proved that $E[X] = \mu$.

We want to find its variance, $Var(X)$

- We let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$Var(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}} dy$$

Normal random variables.

Example (continued). We let $y = \frac{x-\mu}{\sigma}$, then $dx = \sigma dy$ and

$$\text{Var}(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}} dy$$

Now, let $u(y) = y$ and $v(y) = -e^{-\frac{y^2}{2}}$. Then $u'(y) = 1$ and $v'(y) = y \cdot e^{-\frac{y^2}{2}}$, and $\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{\infty} u(y)v'(y)dy$

Integrating by parts, we obtain

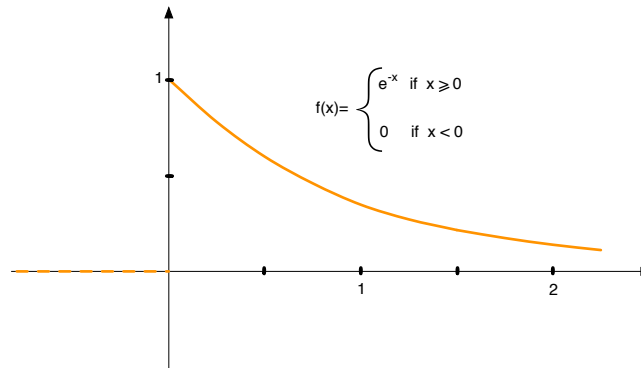
$$\begin{aligned} \text{Var}(X) &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}} dy = \frac{\sigma^2}{\sqrt{2\pi}} \left[-ye^{-\frac{y^2}{2}} \right]_{-\infty}^{\infty} - \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-e^{-\frac{y^2}{2}} \right) dy \\ &= \sigma^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sigma^2 \end{aligned}$$

Expectation and variance.

- Given $\lambda > 0$. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then X is said to be an **exponential random variable** with parameter λ .

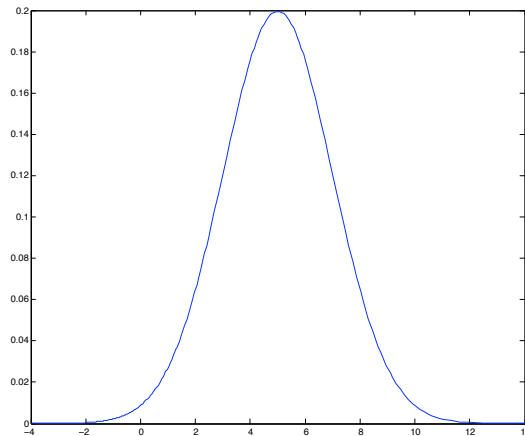


Here $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$

Expectation and variance.

X is a **Normal (Gaussian) random variable** with parameters μ and σ^2 if its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$



Here $E[X] = \mu$ and $Var(X) = \sigma^2$