MTH 361

Lectures 9 - 13

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Topics:

- Introduction to random variables.
- Binomial random variables.
- Expectation of a discrete random variable.
- Poisson random variables.
- Poisson vs Binomial.
- Geometric random variables.

- Examples with discrete random variables.
- Variance and standard deviation.
- Markov inequality.
- Chebyshev inequality.
- Review and examples.

Introduction to random variables.

Consider a sample space $\mathcal{S}$ and a probability function P.

- **Definition.** A function from $\mathcal{S}$ to $\mathbb{R}$ is a **random variable**.

- **Example.** Roll two fair dice. Let $X(i,j) = i + j$ for each outcome (i,j) in $\mathcal{S}$. Then X is a random variable representing the sum of the digits on the dice.

X :

1,1	1,2	1,3	1,4	1,5	1, 6
2,1	2,2	2,3	2,4	2, 5	2,6
3,1	3,2	3,3	3, 4	3,5	3,6
4,1	4,2	4, 3	4,4	4,5	4,6
5,1	5, 2	5,3	5,4	5,5	5,6
6, 1	6,2	6,3	6,4	6,5	6,6

$$\longrightarrow \left\{ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} \right\}$$

Introduction to random variables.

- **Example.** Roll two fair dice. Let $X(i,j) = i + j$ for each outcome (i,j) in $\mathcal{S}$. Then X is a random variable representing the sum of the digits on the dice.

$$X : \begin{array}{|c|c|c|c|c|c|} \hline 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ \hline 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ \hline 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ \hline 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ \hline 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ \hline 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \\ \hline \end{array} \longrightarrow \left\{ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \right\}$$

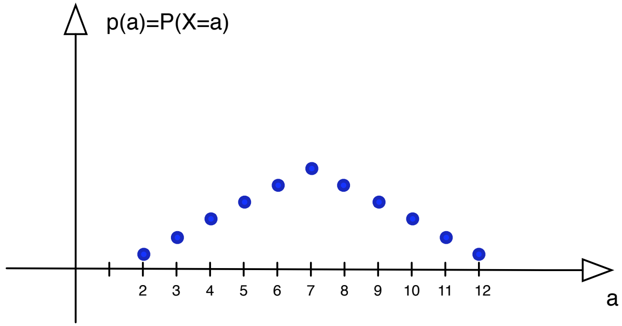
Here, for example, $X(3,1) = 4$ and $X(5,6) = 11$.

We are interested in finding the following probabilities:

$$p(a) = P(X = a) \quad \text{for} \quad a = 2, 3, \ldots, 12$$

Introduction to random variables.

- **Example.** Roll two fair dice. Let $X(i,j) = i + j$ for each outcome (i,j) in $\mathcal{S}$. Then X is a random variable representing the sum of the digits on the dice.



We are interested in finding the following probabilities:

$$p(a) = P(X = a) \quad \text{for} \quad a = 2, 3, \ldots, 12$$

$$p(2) = \tfrac{1}{36},\ p(3) = \tfrac{2}{36},\ p(4) = \tfrac{3}{36},\ p(5) = \tfrac{4}{36},\ p(6) = \tfrac{5}{36},\ p(7) = \tfrac{6}{36}$$

$$p(8) = \tfrac{5}{36},\ p(9) = \tfrac{4}{36},\ p(10) = \tfrac{3}{36},\ p(11) = \tfrac{2}{36},\ p(12) = \tfrac{1}{36}$$

Introduction to random variables.

Let X a discrete random variable. That is X assumes a discrete (countable) number of values.

- **Definition.** Function $p(a) = P(X = a)$ is called the probability mass function (or distribution function).

- **Definition.** Function $F(a) = P(X \leq a)$ is called the cumulative distribution function.

- **Note.** $$\sum_{a:\ p(a)>0} p(a) = 1$$

  In the previous example, $p(2)+p(3)+\cdots+p(12) = 1$.

- **Note.** $0 \leq F(a) \leq 1$

- **Note.** $F(a) = \sum_{x:\ x \leq a} p(x)$

Bernoulli trials and Bernoulli random variables.

For a given $0 \leq p \leq 1$, a Bernoulli trial is an experiment with exactly two possible outcomes, success and failure, in which the probability of success is p and probability of failure is $1 - p$.

Here, the sample space $\mathcal{S}$ consists of the two outcomes, success and failure, and

$$P(\text{success}) = p \quad \text{and} \quad P(\text{failure}) = 1 - p$$

Bernoulli random variable X with parameter p counts the number of successes after one Bernoulli trial, and thus,

$$P(X = 1) = p \quad \text{and} \quad P(X = 0) = 1 - p$$

Expectation of a discrete random variable.

• **Definition.** Let X be a discrete random variable with the probability mass function $p(x)$. Then its expected value is

$$E[X] = \sum_{x:\, p(x)>0} x \cdot p(x)$$

• **Example.** Let X be a Bernoulli random variable with parameter p. Then

$p(1) = P(X = 1) = p$ and $p(0) = P(X = 0) = 1-p$

and

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1) = p$$

• **Example.** Roll two fair dice. Let X represent the sum of the digits on the dice. Then

$$E[X] = 2 \cdot p(2) + 3 \cdot p(3) + \cdots + 12 \cdot p(12) = \frac{252}{36} = 7$$

Expectation of a discrete random variable.

• **Definition.** Let X be a discrete random variable with the probability mass function $p(x)$. Then its **expected value** is

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• **Example.** Roll two fair dice. Let X represent the sum of the digits on the dice. Then

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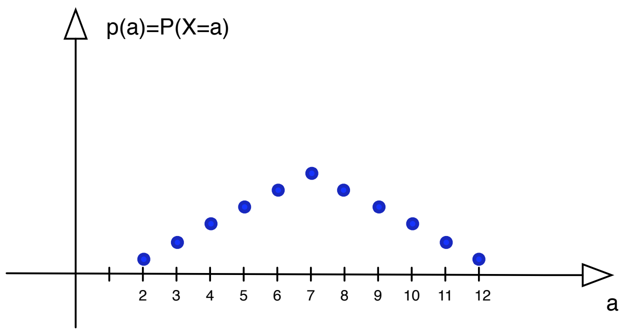


This corresponds to a **center of mass** of $p(a)$.

Binomial random variable. Recall the following example.

• **Example.** Consider performing independent Bernoulli trials, each with probability p of success and probability $1-p$ of failure. Let X be a random variable representing the number of successes in n Bernoulli trials. Find $P(X=k)$ for $k=0,1,\ldots,n$.

• **Solution.**

Each outcome with k successes and $n-k$ failures, its probability

$$P(\underbrace{SFSS\ldots FFS}_{k\ S's \text{ and } n-k\ F's}) = p^k(1-p)^{n-k}$$

and $$P(X=k) = \binom{n}{k}p^k(1-p)^{n-k} \quad \text{for each } k=0,1,\ldots,n$$

because there are $\binom{n}{k}$ such outcomes.

• **Definition.** The random variable X in the above example is the binomial random variable with parameters (n,p).

Check: $\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k}p^k(1-p)^{n-k} = \big(p+(1-p)\big)^n = 1^n = 1.$

Expectation of a discrete random variable.

Let X be a binomial random variable with parameters (n, p). Then its probability mass function is known to be

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for each } k = 0, 1, \ldots, n$$

• **Definition.** Let X be a discrete random variable with the probability mass function $p(x)$. Then its expected value is

$$E[X] = \sum_{x:\, p(x)>0} x \cdot p(x)$$

• **Example.** Let X be a binomial random variable with parameters (n, p). Then

$$E[X] = \sum_{k=0}^{n} k \cdot p(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = ?$$

Expectation of a discrete random variable.

• **Example.** Let X be a **binomial random variable** with parameters (n, p). Then $E[X] = np$ since

$$E[X] = \sum_{k=0}^{n} k \cdot p(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \, p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} \, p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \, p^k (1-p)^{n-k}$$

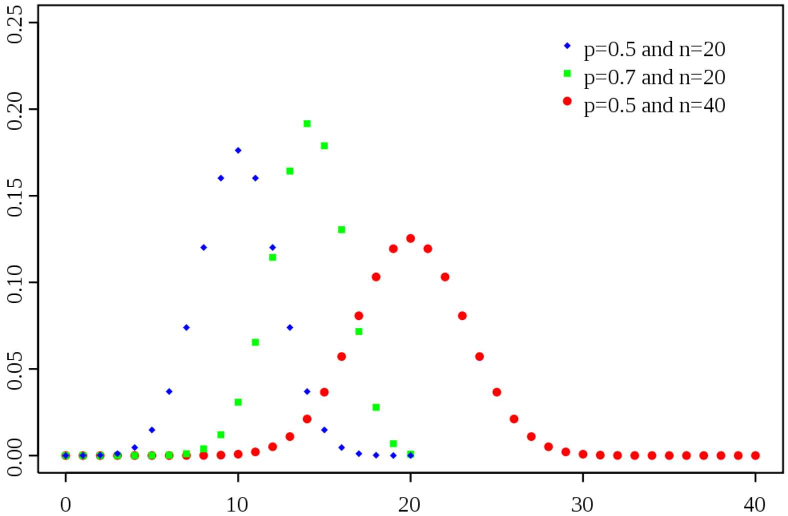
$$= \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} \, p^{j+1} (1-p)^{n-1-j} = np \cdot \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \, p^j (1-p)^{n-1-j},$$

where the new index $j = k - 1$. Thus

$$E[X] = np \cdot \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \, p^j (1-p)^{n-1-j} = np \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} \, p^j (1-p)^{n-1-j}$$

$$= np \cdot (p + (1-p))^{n-1} = np \qquad \text{by the Binomial theorem}$$

Binomial random variable.



Picture credit: Wikipedia.org

$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for each $k = 0, 1, \ldots, n$ and $E[X] = np$

Poisson random variable.

- **Recall that** $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$ for all $-\infty < a < +\infty$

- **Definition.** Let $\lambda > 0$. A discrete random variable such that its probability mass function

$$p(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \qquad \text{for each } k = 0, 1, \ldots$$

is a **Poisson random variable** with parameter $\lambda > 0$.

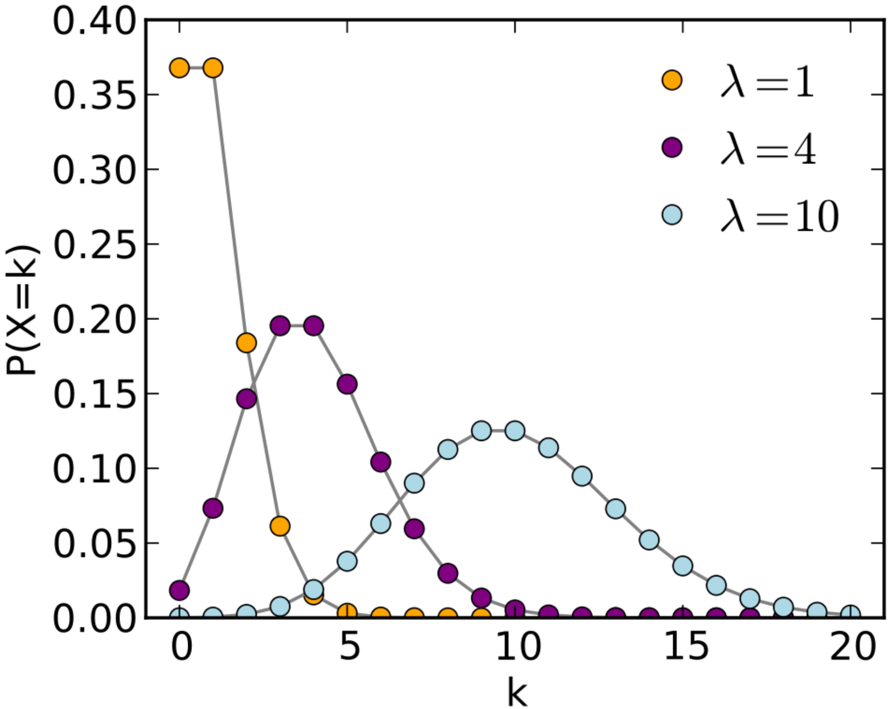
- Function $p(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ is **Poisson distribution**

- Check $\sum_{k=0}^{\infty} p(k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$

- **Expectation:** Let X be a Poisson random variable with parameter λ. Then $E[X] = \lambda$ since

$$E[X] = \sum_{k=0}^{\infty} k \cdot p(k) = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} = \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda$$

Poisson random variable.



Picture credit: Wikipedia.org

$p(k) = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$ for each $k = 0, 1, \ldots$ and $E[X] = \lambda$

Poisson vs Binomial.

Let $\lambda > 0$ be given. Suppose Y is a Poisson random variable with parameter λ. Then its probability mass function

$$P(Y = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \text{for each } k = 0, 1, \ldots$$

Now, let S_n be a Binomial random variable with parameters n and $p = \frac{\lambda}{n}$. Then its probability mass function

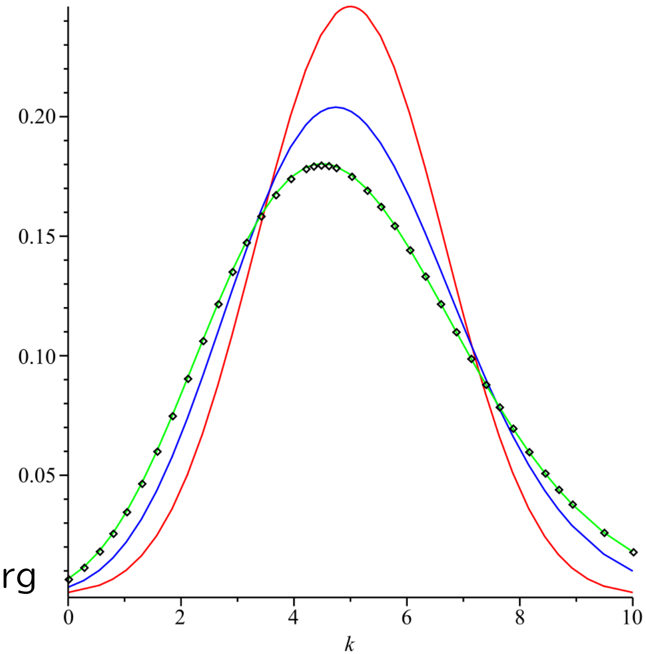
$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \quad \text{for each } k = 0, 1, \ldots, n$$

- **Theorem.** Consider integer $k \geq 0$. Then for n large enough,

$$P(S_n = k) \approx P(Y = k)$$

Namely, $\lim_{n \to \infty} P(S_n = k) = P(Y = k)$

Poisson vs Binomial.



Picture credit: Wikipedia.org

Dots: Poisson($\lambda = 5$) Red: Binomial($n = 10,\ p = \frac{1}{2}$)

Blue: Binomial($n = 20,\ p = \frac{1}{4}$)

Green: Binomial($n = 1000,\ p = \frac{1}{200}$)

Poisson vs Binomial.

Let $\lambda > 0$ be given. Suppose Y is a Poisson random variable with parameter λ and S_n is a Binomial random variable with parameters n and $p = \frac{\lambda}{n}$.

• **Theorem.** For a given integer $k \geq 0$, $\lim_{n\to\infty} P(S_n = k) = P(Y = k)$.

Thus, for n large enough, $P(S_n = k) \approx P(Y = k)$.

Proof: k is fixed, and

$$P(S_n = k) = \frac{n!}{k!(n-k)!} \cdot \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{n^k} \cdot \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$= \frac{\lambda^k}{k!} \cdot \frac{(n-k+1)(n-k+2)\ldots n}{n^k} \cdot \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k} \longrightarrow e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \text{as } n \to \infty$$

since $\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$, $\frac{(n-k+1)(n-k+2)\ldots n}{n^k} \to 1$, and $\left(1 - \frac{\lambda}{n}\right)^k \to 1^k = 1$.

Geometric random variables.

• **Example.** Consider performing independent Bernoulli trials, each with probability p of success and probability $1-p$ of failure. Let X be a random variable representing the number of trials until the first success. Find $P(X=k)$ for $k=1,2,\ldots$.

• **Solution.**

The sample space $\mathcal{S}$ consists of the outcomes of infinitely many Bernoulli trials. For example $FFSFSSFS\ldots$ is one such outcome. Here X is a function from the sample space $\mathcal{S}$ to $\mathbb{R}$, and here

$$X(FFSFSSFS\ldots) = 3$$

$$P(X=3) = P(F_1F_2S_3) = P(F_1)\cdot P(F_2)\cdot P(S_3) = p\cdot(1-p)^2$$

and $$P(X=k) = P(F_1)\cdot\cdots\cdot P(F_{k-1})\cdot P(S_k) = p\cdot(1-p)^{k-1} \quad \text{for each } k=1,2,\ldots$$

• **Definition.** The random variable X in the above example is called a geometric random variable with parameter p.

Geometric random variables.

A **geometric random variable** with parameter p is characterized by a probability mass function,

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

- We need to check that $\sum_{k=1}^{\infty} p(k) = 1$.

Geometric series: $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$

Claim: For $x \neq 1$, $\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$

Proof: $(1-x)(1+x+x^2+x^3+\cdots+x^n) = [1+x+x^2+x^3+\cdots+x^n] - [x+x^2+x^3+\cdots+x^n+x^{n+1}] = 1-x^{n+1}$

Summing the geometric series: For $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \left(\sum_{k=0}^{n} x^k \right) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}$$

Geometric random variables.

Summing the geometric series: For $|x| < 1$,

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A **geometric random variable** with parameter p is characterized by a probability mass function,

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

- We need to check that $\sum_{k=1}^{\infty} p(k) = 1$.

$$\sum_{k=1}^{\infty} p(k) = p \cdot \sum_{k=1}^{\infty} (1-p)^{k-1} = p \cdot \sum_{j=0}^{\infty} (1-p)^j = p \cdot \frac{1}{1-(1-p)} = 1,$$

where $j = k - 1$.

Geometric random variables.

A **geometric random variable** with parameter p is characterized by a probability mass function,

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

We need to find its expectation $E[X] = \sum_{k=1}^{\infty} k \cdot p(k)$.

$$E[X] = \sum_{k=1}^{\infty} k \cdot p(k) = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = ?$$

Geometric random variables.

A **geometric random variable** with parameter p is characterized by a probability mass function,

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

We need to find its expectation $E[X] = \sum_{k=1}^{\infty} k \cdot p(k)$.

$$E[X] = \sum_{k=1}^{\infty} k \cdot p(k) = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = ?$$

Here $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$ as

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (x^k)' = \left(\sum_{k=0}^{\infty} x^k\right)' = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

and therefore $E[X] = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}$

Geometric random variables.

Alternative proof of $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$.

$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots =$

$$
\begin{array}{cccccccccccccc}
1 + & x + & x^2 + & x^3 + & x^4 + \cdots = \frac{1}{1-x} \\
 & + & + & + & + \\
 & x + & x^2 + & x^3 + & x^4 + \cdots = \frac{x}{1-x} \\
 & & + & + & + \\
 & & x^2 + & x^3 + & x^4 + \cdots = \frac{x^2}{1-x} \\
 & & & + & + \\
 & & & x^3 + & x^4 + \cdots = \frac{x^3}{1-x} \\
 & & & & + \\
 & & & & x^4 + \dots \quad \vdots
\end{array}
$$

$\frac{1}{1-x} + \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} + \cdots = \frac{1}{1-x} \cdot (1 + x + x^2 + \dots) = \frac{1}{(1-x)^2}$

Geometric random variables.

Let X be **geometric random variable** with parameter p. Then

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

and

$$E[X] = \frac{1}{p}$$

- Prove the following *memorylessensss* property:

$$P(X = n + k \mid X > n) = P(X = k)$$

for any two positive integers n and k.

We observe that here, $P(X = n + k \mid X > n) = \frac{P(X=n+k)}{P(X>n)}$

- Apply the above to coin tossing. Give an example.

Geometric random variables.

Let X be **geometric random variable** with parameter p. Then

$$p(k) = p \cdot (1-p)^{k-1} \quad \text{for each } k = 1, 2, \ldots$$

and

$$E[X] = \frac{1}{p}$$

- **Example.** Find probability $P(X \geq 10)$.

- **Example.** Let $p = \frac{1}{2}$. Find probability $P(X \geq 20)$.

Discrete random variables.

• **Example.** Let X be a Binomial random variable with parameters $n = 200$ and $p = 0.035$. Find probabilities $P(X = 4)$ and $P(X = 6)$.

Here $P(X = 4) = \binom{200}{4}(0.035)^4(0.965)^{196} = 0.09003862196\ldots$
and $P(X = 6) = \binom{200}{6}(0.035)^6(0.965)^{194} = 0.1508966957\ldots$

• **Example.** Let X be a Poisson random variable with parameter $\lambda = 7$. Find probabilities $P(X = 4)$ and $P(X = 6)$.

Here $P(X = 4) = e^{-7} \cdot \frac{7^4}{4!} = 0.09122619167\ldots$
and $P(X = 6) = e^{-7} \cdot \frac{7^6}{6!} = 0.1490027797\ldots$

• **Example.** Let X be a geometric random variable with parameter $p = \frac{1}{7}$. Find probabilities $P(X = 4)$ and $P(X = 6)$.

Here $P(X = 4) = \frac{1}{7} \cdot \frac{6^3}{7^3} = 0.08996251562\ldots$
and $P(X = 6) = \frac{1}{7} \cdot \frac{6^5}{7^5} = \frac{7776}{117649} = 0.06609490943\ldots$

Variance and standard deviation.

• **Theorem.** Let X be a discrete random variable characterized by its probability mass function $p(x)$. Then, for any real valued function g, $g(X)$ will also be a **random variable**, and

$$E[g(X)] = \sum_{x:\, p(x)>0} g(x)\, p(x)$$

• **Example.** We roll a fair die once, and square the outcome. Let X be a random variable representing the outcome. Then $Y = X^2$ will be a random variable representing the square of the outcome. Here

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = p_X(5) = p_X(6) = \frac{1}{6}$$

will be the probability mass function for X, and

$$p_Y(1) = p_Y(4) = p_Y(9) = p_Y(16) = p_Y(25) = p_Y(36) = \frac{1}{6}$$

will be the probability mass function for Y. Then

$$E[Y] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}$$

Variance and standard deviation.

• **Theorem.** Let X be a discrete random variable characterized by its probability mass function $p_X(x)$. Then, for any real valued function g, $g(X)$ will also be a **random variable**, and

$$E[g(X)] = \sum_{x:\, p_X(x)>0} g(x)\, p_X(x)$$

Proof: Let $Y = g(X)$. We find the probability mass function $p_Y(y)$ of Y:

$$p_Y(y) = P\big(g(X) = y\big) = \sum_{x:\, g(x)=y} P(X = x) = \sum_{x:\, g(x)=y} p_X(x)$$

as $\big\{g(X) = y\big\} = \bigcup_{x:\, g(x)=y} \{X = x\}$ is a union of disjoint events.

Thus, $$E[Y] = \sum_y y\, p_Y(y) = \sum_y \left(y \sum_{x:\, g(x)=y} p_X(x) \right) = \sum_y \left(\sum_{x:\, g(x)=y} y\, p_X(x) \right)$$

$$= \sum_y \left(\sum_{x:\, g(x)=y} g(x)\, p_X(x) \right) = \sum_x g(x)\, p_X(x)$$

Variance and standard deviation.

$$E[g(X)] = \sum_{x:\, p(x)>0} g(x)\, p(x)$$

• **Example.** We roll a fair die once, and square the outcome. Let X be a random variable representing the outcome. Then $Y = X^2$ will be a random variable representing the square of the outcome. Here

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = p_X(5) = p_X(6) = \frac{1}{6}$$

will be the probability mass function for X, and

$$p_Y(1) = p_Y(4) = p_Y(9) = p_Y(16) = p_Y(25) = p_Y(36) = \frac{1}{6}$$

will be the probability mass function for Y. Then

$$E[Y] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}$$

Observe that $\displaystyle\sum_{k=1}^{6} k^2 \cdot p_X(k) = \frac{91}{6}$ as well. Also observe that

$$E[X^2] = \frac{91}{6} \quad \neq \quad \big(E[X]\big)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

Examples.

- **Problem.** Random variable X has the following probability mass function

$$p_X(x) = \begin{cases} \frac{1}{8} & \text{if } x = -2 \\ \frac{5}{8} & \text{if } x = 2 \\ \frac{1}{4} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

That is $p_X(-2) = \frac{1}{8}$, $p_X(2) = \frac{5}{8}$ and $p_X(3) = \frac{1}{4}$.

Compute $E[X]$ and $E[X^2]$. Hint: Recall that $E[g(X)] = \sum_{x:\, p_X(x)>0} g(x)\, p_X(x)$.

Solution: $E[X] = (-2) \cdot p_X(-2) + 2 \cdot p_X(2) + 3 \cdot p_X(3) = \frac{7}{4}$

$$E[X^2] = (-2)^2 \cdot p_X(-2) + 2^2 \cdot p_X(2) + 3^2 \cdot p_X(3) = \frac{21}{4}$$

Examples.

- **Example.** Let X be binomial random variable with parameters $n = 20$ and $p = \frac{1}{4}$. Use the binomial theorem to compute $E[2^X]$.

Solution:

$$E[2^X] = \sum_{k=0}^{n} 2^k \cdot p(k) = \sum_{k=0}^{n} 2^k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \cdot (2p)^k (1-p)^{n-k}$$

$$= (2p + (1-p))^n = (1+p)^n = \left(\frac{5}{4}\right)^{20} = 86.7361738$$

Variance and standard deviation.

- Given constants α and β,

$$E[\alpha X + \beta] = \alpha E[X] + \beta$$

Proof:

$$E[\alpha X+\beta] = \sum_{k:\, p(k)>0} (\alpha k+\beta)\cdot p(k) = \alpha\cdot \sum_{k:\, p(k)>0} kp(k)+\beta\cdot \sum_{k:\, p(k)>0} p(k) = \alpha E[X]+\beta$$

Now, let X be a random variable with mean $E[X] = \mu$.

- **Definition.** The **variance** of X is

$$Var(X) = E\Big[(X-\mu)^2\Big]$$

Note that the variance is a mean square displacement from the mean μ.

- **Definition.** The **standard deviation** of X is

$$SD(X) = \sqrt{Var(X)} = \sqrt{E\Big[(X-\mu)^2\Big]}$$

Variance and standard deviation.

Let X be a random variable with mean $E[X] = \mu$.

• **Definition.** The **standard deviation** of X is

$$SD(X) = \sqrt{Var(X)} = \sqrt{E\left[(X-\mu)^2\right]}$$

Another notation: $\sigma(X)$ and σ.

• **Intuition:** $X = \mu \pm \sigma$

• **Example.** Let X be a Binomial random variable with parameters n and p. We know that $E[X] = np$. It will be shown that the variance

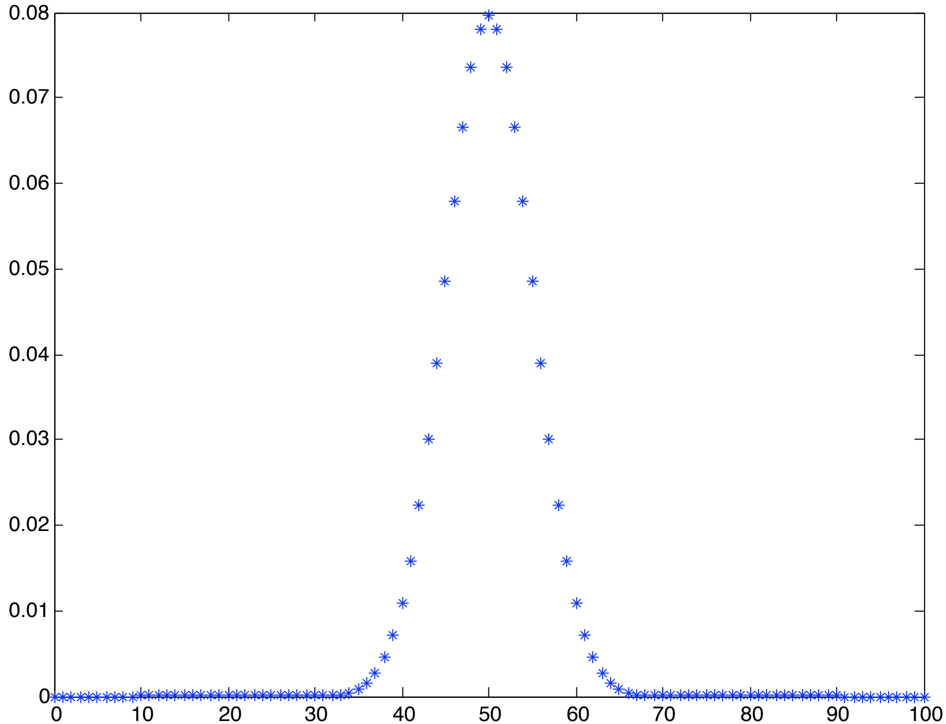
$$Var(X) = np(1-p)$$

Thus

$$X = np \pm \sqrt{np(1-p)}$$

Let X be a Binomial random variable with $n = 100$ and $p = \frac{1}{2}$.

$$X = np \pm \sqrt{np(1-p)} = 50 \pm 5$$



Variance and standard deviation.

Let X be a random variable with mean $E[X] = \mu$.

- **Theorem.** The **variance** of X equals

$$Var(X) = E[X^2] - \mu^2$$

Proof:

$$Var(X) = E\left[(X-\mu)^2\right] = \sum_{a:\, p(a)>0} (a-\mu)^2 \cdot p(a) = \sum_{a:\, p(a)>0} (a^2-2\mu a+\mu^2)\cdot p(a)$$

$$= \sum_{a:\, p(a)>0} a^2 \cdot p(a) - 2\mu \cdot \sum_{a:\, p(a)>0} a \cdot p(a) + \mu^2 \cdot \sum_{a:\, p(a)>0} p(a)$$

$$= \sum_{a:\, p(a)>0} a^2 \cdot p(a) - 2\mu \cdot \mu + \mu^2 \cdot 1 = E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

Variance and standard deviation.

• **Example.** Let X be a Binomial random variable with parameters n and p. Show that

$$Var(X) = np(1-p)$$

Solution: Here $\mu = np$ and

$$Var(X) = E[X^2] - \mu^2 = \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} - \mu^2$$

$$= \sum_{k=0}^{n} (k^2 - k) \cdot \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} - \mu^2$$

$$= \sum_{k=2}^{n} k(k-1) \cdot \binom{n}{k} p^k (1-p)^{n-k} + \mu - \mu^2 = \sum_{k=2}^{n} k(k-1) \cdot \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k} + \mu - \mu^2$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} \cdot p^k (1-p)^{n-k} + \mu - \mu^2$$

Variance and standard deviation.

• **Example.** Let X be a Binomial random variable with parameters n and p. Show that

$$Var(X) = np(1-p)$$

Solution (continued): Here $\mu = np$ and

$$Var(X) = \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} \cdot p^k (1-p)^{n-k} + \mu - \mu^2$$

$$= p^2 \cdot n(n-1) \cdot \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} \cdot p^{k-2}(1-p)^{n-k} + \mu - \mu^2$$

$$= p^2 \cdot n(n-1) \cdot \sum_{j=0}^{n-2} \binom{n-2}{j} \cdot p^j (1-p)^{(n-2)-j} + \mu - \mu^2, \qquad \text{where } j = k-2$$

$$= p^2 \cdot n(n-1) \cdot (p + (1-p))^{n-2} + \mu - \mu^2 = p^2 \cdot n(n-1) + \mu - \mu^2$$

$$= p^2 \cdot (n^2 - n) + np - (np)^2 = -np^2 + np = np(1-p)$$

Variance and standard deviation.

• **Example.** Let X be a Poisson random variable with parameter $\lambda > 0$. Show that

$$Var(X) = \lambda$$

Solution: Here $\mu = \lambda$ and

$$Var(X) = E[X^2] - \mu^2 = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda}\frac{\lambda^k}{k!} - \mu^2$$

$$= \sum_{k=0}^{\infty}(k^2 - k) \cdot e^{-\lambda}\frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} k \cdot e^{-\lambda}\frac{\lambda^k}{k!} - \mu^2$$

$$= \sum_{k=2}^{\infty} k(k-1) \cdot e^{-\lambda}\frac{\lambda^k}{k!} + \mu - \mu^2 = \sum_{k=2}^{\infty} e^{-\lambda}\frac{\lambda^k}{(k-2)!} + \mu - \mu^2$$

$$= \sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^{j+2}}{j!} + \mu - \mu^2 = \lambda^2 \cdot e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \mu - \mu^2, \qquad \text{where } j = k-2$$

$$= \lambda^2 \cdot e^{-\lambda} \cdot e^{\lambda} + \lambda - \lambda^2 = \lambda$$

Variance and standard deviation.

- **Example.** Let X be a geometric random variable with parameter p. Show that

$$Var(X) = \frac{1-p}{p^2}$$

Solution: Here $\mu = \frac{1}{p}$ and

$$Var(X) = E[X^2] - \mu^2 = \sum_{k=1}^{\infty} k^2 \cdot p \cdot (1-p)^{k-1} - \mu^2$$

$$= \sum_{k=1}^{\infty} k(k-1) \cdot p \cdot (1-p)^{k-1} + \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} - \mu^2$$

$$= p \cdot (1-p) \cdot \sum_{k=0}^{\infty} k(k-1) \cdot (1-p)^{k-2} + \mu - \mu^2$$

Variance and standard deviation.

- **Example.** Let X be a geometric random variable with parameter p. Show that

$$Var(X) = \frac{1-p}{p^2}$$

Solution (continued): Here $\mu = \frac{1}{p}$ and

$$Var(X) = p \cdot (1-p) \cdot \sum_{k=0}^{\infty} k(k-1) \cdot (1-p)^{k-2} + \mu - \mu^2$$

Now, for $|x| < 1$,

$$\sum_{k=0}^{\infty} k(k-1) \cdot x^{k-2} = \sum_{k=0}^{\infty} \left(x^k\right)'' = \frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3}$$

Hence,

$$Var(X) = p \cdot (1-p) \cdot \frac{2}{p^3} + \mu - \mu^2 = 2 \cdot \frac{1-p}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Variance and standard deviation.

- **Theorem.** The **variance** of X equals

$$Var(X) = E[X^2] - \mu^2$$

- **Example.** Let X be a Binomial random variable with parameters n and p. Then

$$Var(X) = np(1-p)$$

- **Example.** Let X be a Poisson random variable with parameter $\lambda > 0$. Then

$$Var(X) = \lambda$$

- **Example.** Let X be a geometric random variable with parameter p. Then

$$Var(X) = \frac{1-p}{p^2}$$

Markov inequality.

• **Example.** When a certain lab experiment is performed, the outcome is an integer number on the scale from 0 to $20,000$. Analyzing the outcomes of multiple identical experiments performed independently of each other it was noticed that the average value stays around 440. Suppose the threshold value is $10,000$. If this is all we know, can we estimate how small is the probability that the outcome of one such experiment yields a value greater or equal to $10,000$.

Same stated in terms of random variables: Let X be a random variable, taking integer values from 0 to $20,000$. We don't know its probability mass function $p(k)$ $(k = 0, 1, 2, \ldots, 20K)$. However we know that its expectation $E[X] = 440$. What can we say about the probability of going above the threshold

$$P(X \geq 10,000) \qquad ?$$

Can we bound it?

Markov inequality.

Same stated in terms of random variables: Let X be a random variable, taking integer values from 0 to $20,000$. We don't know its probability mass function $p(k)$ $(k = 0, 1, 2, \ldots, 20K)$. However we know that its expectation $E[X] = 440$. What can we say about the probability of going above the threshold

$$P(X \geq 10,000) \qquad ?$$

Can we bound it?

Theorem. (Markov inequality.) If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

$$P(X \geq \alpha) \; \leq \; \frac{E[X]}{\alpha}$$

Solution to the above example:

$$P(X \geq 10,000) \; \leq \; \frac{440}{10,000} = 0.044$$

Markov inequality.

Theorem. (Markov inequality.) If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

$$P(X \geq \alpha) \ \leq \ \frac{E[X]}{\alpha}$$

Proof:

$$P(X \geq \alpha) = \sum_{k:\, k \geq \alpha} p(k) \ \leq \ \sum_{k:\, k \geq \alpha} \frac{k}{\alpha} \cdot p(k) = \frac{1}{\alpha} \cdot \sum_{k:\, k \geq \alpha} k \cdot p(k) \ \leq \ \frac{1}{\alpha} \cdot \sum_{k:\, k \geq 0} k \cdot p(k) \ = \frac{E[X]}{\alpha}$$

• **Example.** Let X be a Binomial random variable with parameters $n = 2,500$ and $p = 0.2$. Use Markov inequality to give an upper bound on the following probability

$$P(X \geq 540) = \sum_{k=540}^{2,500} \binom{2500}{k} \cdot (0.2)^k \cdot (0.8)^{2,500-k}$$

Markov inequality.

Theorem. (Markov inequality.) If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

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• **Example.** Let X be a Binomial random variable with parameters $n = 2,500$ and $p = 0.2$. Use Markov inequality to give an upper bound on the following probability

$$P(X \geq 540) = \sum_{k=540}^{2{,}500} \binom{2500}{k} \cdot (0.2)^k \cdot (0.8)^{2{,}500-k}$$

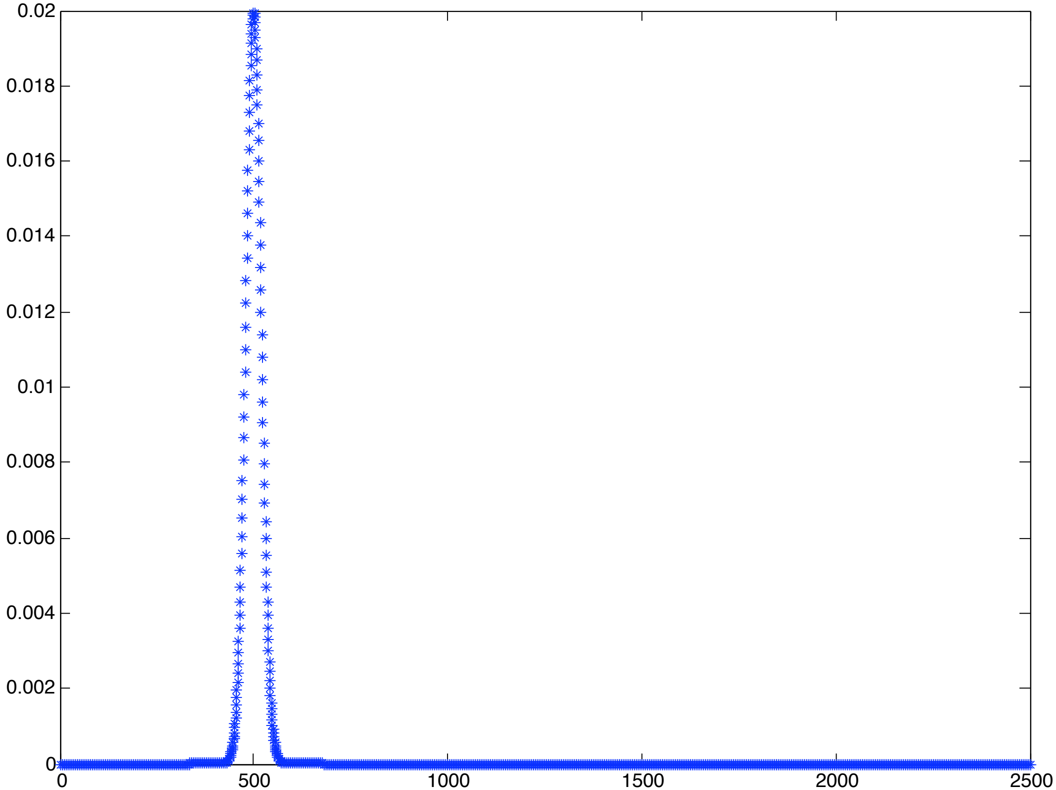
Solution: Here $E[X] = np = 500$. Thus

$$P(X \geq 540) \;\leq\; \frac{500}{540} = 0.\overline{925}\ldots$$

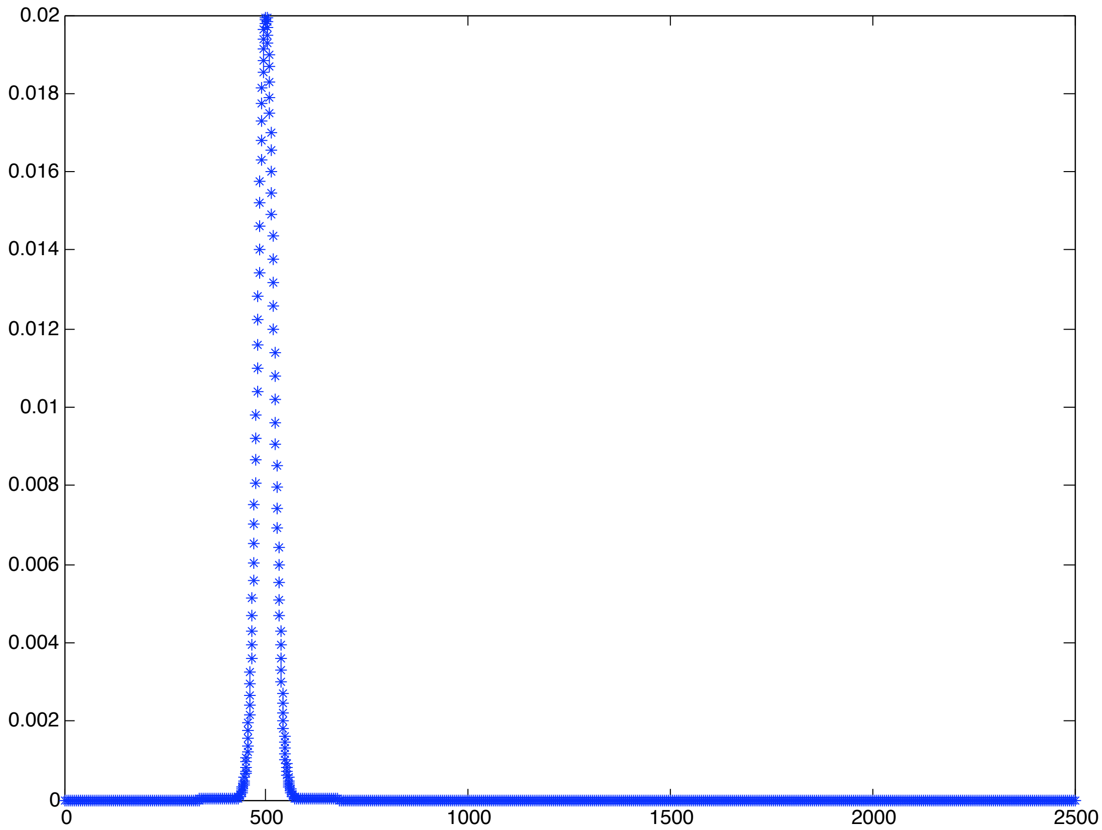
• **Comment:** Here we also know the standard deviation $\sigma = \sqrt{np(1-p)} = 20$.

Thus we know that $X = \mu \pm \sigma = 500 \pm 20$, making us believe that $P(X \geq 540)$ is **much** smaller than 92.5%.

We know that $X = \mu \pm \sigma = 500 \pm 20$, making us believe that $P(X \geq 540)$ is **much** smaller than 92.5%.



In fact, $P(X \geq 540) \approx 0.0249 \;<<\; 0.\overline{925}$.



Chebyshev inequality.

Theorem. (Chebyshev inequality.) If X is a random variable with finite mean μ and variance, then for any $\kappa > 0$,

$$P(|X-\mu| \geq \kappa) \;\leq\; \frac{Var(X)}{\kappa^2}$$

• **Example.** Let X be a Binomial random variable with parameters $n = 2,500$ and $p = 0.2$. Give an upper bound on the following probability

$$P(X \geq 540) = \sum_{k=540}^{2{,}500} \binom{2500}{k} \cdot (0.2)^k \cdot (0.8)^{2{,}500-k}$$

Solution: Here $\mu = np = 500$ and $Var(X) = np(1-p) = 400$. Thus

$$P(X \geq 540) = P(X-\mu \geq 40) \;\leq\; P(|X-\mu| \geq 40) \;\leq\; \frac{400}{40^2} = 0.25$$

Markov and Chebyshev inequalities.

Theorem. (Markov inequality.) If X is a random variable that takes only nonnegative values, then for any $\alpha > 0$,

$$P(X \geq \alpha) \ \leq \ \frac{E[X]}{\alpha}$$

Theorem. (Chebyshev inequality.) If X is a random variable with finite mean μ and variance, then for any $\kappa > 0$,

$$P(|X - \mu| \geq \kappa) \ \leq \ \frac{Var(X)}{\kappa^2}$$

Proof: Let $Y = (X - \mu)^2$, then $E[Y] = Var(X)$ and

$$P(|X-\mu| \geq \kappa) = P((X-\mu)^2 \geq \kappa^2) = P(Y \geq \kappa^2) \ \leq \ \frac{E[Y]}{\kappa^2} = \frac{Var(X)}{\kappa^2}$$

using Markov inequality for Y, since Y is a nonnegative random variable.

Review problems.

• **Problem.** Prove that if $P(A) = P(B) = \frac{3}{4}$, then $P(A|B) \geq \frac{2}{3}$.
Hint: Use the inclusion-exclusion formula.

Solution:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) + P(B) - P(A \cup B)}{P(B)}$$

$$= 2 - \frac{P(A \cup B)}{3/4} \geq 2 - \frac{1}{3/4} = 2 - \frac{4}{3} = \frac{2}{3}$$

as $\frac{P(A \cup B)}{3/4} \leq \frac{1}{3/4}$

Review problems.

• **Problem.** Let $S = \{a, b, c\}$ be the sample space for the experiment with positive probabilities for each outcome. Given three events $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, and $B = \{b\}$. Check if the following is true regardless of the probability values for the outcomes:

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$$

Solution: $A_1 \cup A_2 = S = \{a, b, c\}$

Hence,

$$P(A_1 \cup A_2|B) = P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

While,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

and

$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Hence $P(A_1 \cup A_2|B) \neq P(A_1|B) + P(A_2|B)$

Review problems.

• **Problem.** Consider two urns, one containing 1 black and 7 red marble, the other containing 6 black and 1 red marble. An urn is selected at random, and a marble is drawn at random from the selected urn. What is the probability that the first urn was the one selected, given that the marble is red?
Hint: use Bayes' formula, $P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F)+P(E|\overline{F})P(\overline{F})}$

Solution: Let $U_1 = \{\text{Urn 1 selected}\}$, $U_2 = \{\text{Urn 2 selected}\}$, and $R = \{\text{red marble selected}\}$. Then

$$P(R) = P(R \cap U_1) + P(R \cap U_2) = P(R|U_1)P(U_1) + P(R|U_2)P(U_2) = \frac{7}{8} \cdot \frac{1}{2} + \frac{1}{7} \cdot \frac{1}{2} = \frac{57}{112}$$

$$P(U_1|R) = \frac{P(R \cap U_1)}{P(R)} = \frac{P(R|U_1)P(U_1)}{P(R|U_1)P(U_1) + P(R|U_2)P(U_2)} = \frac{7/16}{57/112} = \frac{49}{57}$$

Review problems.

• **Problem.** If 17 people are to be divided into two committees of respective sizes 5, and 12, how many divisions are possible? Here each person can serve only on one committee.

Solution: We are splitting 17 people into two groups: committee A of five and committee B of 12. Since the 17 people are different individuals, it is the same as making 17-long strings of 5 A's and 12 B's. Thus there are

$$\binom{17}{5}$$

ways to do so.

Review problems.

- **Problem.** Use the Binomial Theorem to show

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

Solution: Recall the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging in $x = -1$ and $y = 1$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n = (-1+1)^n = 0$$

Review problems.

- **Problem.** Compute $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$

Solution: Recall the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging in $x = 1$ and $y = 2$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n = (1+2)^n = 3^n$$

Review problems.

• **Problem.** If a fair die is rolled five times, what is the probability that 6 comes up exactly three times?

Solution: Since this is a fair die, the probability that 6 comes up when the die is rolled is

$$p = \frac{1}{6}$$

So, we perform $n = 5$ independent experiments with probability $p = \frac{1}{6}$ of success. Then the probability of exactly $k = 3$ successes in the $n = 5$ trials is Binomial

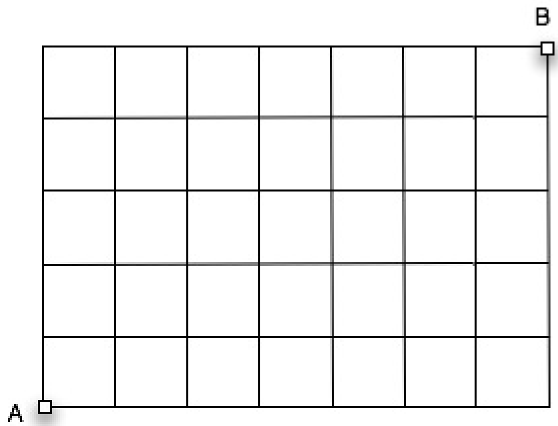
$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} = \binom{5}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2$$

$$= \binom{5}{3} \cdot \frac{25}{6^5} = \frac{125}{3,888} = 0.03\overline{2150}$$

Alternatively, it can be solved via counting.

Review problems.

• **Problem.** Consider a walk on the grid pictured below, originating at the point labelled **A**. Each time the walker can go one step up or one step to the right . This is continued until the point labeled **B** is reached. How many different paths from **A** to **B** are possible? Here is an example of such path: `Up-Up-Right-Right-Right-Up-Right-Right-Up-Right-Up-Right`



Solution: Each distinct path corresponds to a distinct string made of 5 U's and 7 R's, where U and R stand for `Up` and `Right` respectively. Thus there are $\binom{12}{5}$ such paths.

Review problems.

• **Example: Birthday Problem.** Find the probability that among n persons, at least two have birthdays on the same day (but not necessarily in the same year). Assume all days of the year are equally likely to be one's birthday, and ignore February 29th.

Solution: There are two cases: when $n \leq 365$ and when $n > 365$.

If $n \leq 365$, then we find the probability that they all have different birthdays: $\frac{365 \cdot 364 \ldots (365-n+1)}{365^n}$

and subtract it from 1, obtaining

$$1 - \frac{365 \cdot 364 \ldots (365 - n + 1)}{365^n}$$

When $n > 365$, this probability is equal to 1.