1. Let $X$ be a continuous random variable with probability density function

$$f(x) = \begin{cases} 
\frac{8}{x^3} & \text{if } x \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

Check that $f(x)$ is indeed a probability density function. Find $P(X > 5)$ and $E[X]$.

2. Let $X$ be a continuous random variable with probability density function

$$f(x) = \begin{cases} 
c(x-1)^4 & \text{if } 1 < x < 2, \\
0 & \text{otherwise}
\end{cases}$$

where $c$ is a constant. Find $c$, and $E[X]$.

3. Let $X$ be a continuous random variable with probability density function

$$f(x) = \begin{cases} 
ax^2 + bx & \text{for } 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

where $a$ and $b$ are constants. If $E[X] = 0.75$, find $a$, $b$, and $Var(X)$.

4. Show that an exponential random variable is memoryless. That is, if $X$ is exponential with parameter $\lambda > 0$, then

$$P(X > s + t \mid X > s) = P(X > t) \quad \text{for } s, t \geq 0$$

Hint: see example 5.1 in the book.

5. Let $f(t)$ be the probability density function, and $F(t)$ be the corresponding cumulative distribution function. Define the hazard function $h(t) = \frac{f(t)}{1-F(t)}$. Show that if $X$ is an exponential random variable with parameter $\lambda > 0$, then its hazard function will be a constant

$$h(t) = \lambda$$

for all $t > 0$. Think of how this relates to the memorylessness property of exponential random variables.

6. Let $X$ be an exponential random variable with parameter $\lambda = 2$. Compute $E[e^X]$.

7. Consider a random variable $X$ with $E[X] = \mu$ and $Var(X) = \sigma^2$. Let $Y = \frac{X-\mu}{\sigma}$. Find $E[Y]$ and $Var(Y)$. The answer should not depend on whether $X$ is a discrete or continuous random variable.
8. We say that two discrete random variables \( X \) and \( Y \), are independent when

\[
P(X = a, Y = b) = P(X = a)P(Y = b)
\]

for all \( a \) and \( b \) in the corresponding sample spaces.

Let \( X_1 \) and \( X_2 \) be independent Poisson random variables with parameters \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \) respectively. Find the probability of the event that \( X_1 + X_2 = 3 \).

Hint: Since

\[
\{X_1 + X_2 = 3\} = \{X_1 = 0, X_2 = 3\} \cup \{X_1 = 1, X_2 = 2\} \cup \{X_1 = 2, X_2 = 1\} \cup \{X_1 = 3, X_2 = 0\}
\]

is the union of disjoint (mutually exclusive) events,

\[
P(X_1 + X_2 = 3) = P(X_1 = 0, X_2 = 3) + P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) + P(X_1 = 3, X_2 = 0).
\]