Topics:

- Sets. Venn diagrams.
- Predicate calculus. Truth tables.
- Introduction to discrete probability.
- Probability by counting.
- Properties of probability function.
- Inclusion-exclusion formula of probability.
SETS: notions and examples.

• a set is a collection of objects (elements).

• Integers: \( \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \} \)

• Rational numbers:
  \( \mathbb{Q} = \{ \frac{n}{m} : n \text{ and } m \text{ are integers, and } m \neq 0 \} \)

• Real numbers:
  \( \mathbb{R} = \{ \text{ all values between } -\infty \text{ and } +\infty \} \)

• Empty set \( \emptyset = \{ \} \)
• $U$ is called a *universal set* or a *universe*

• $a \in A$ denotes that $a$ is an element of $A$

• $\overline{A} =$ all elements in the universe $U$ that do not belong to $A$

• *Intersection*: $A \cap B =$ all elements in the universe $U$ that belong to $A$ and $B$

• *Union*: $A \cup B =$ all elements in the universe $U$ that belong to $A$ or $B$, or to both sets, $A$ and $B$
• $A \setminus B$ (or $A - B$) = all elements in the universe $U$ that belong to $A$ but do not belong to $B$

• $A \subseteq B$ ($A$ is a subset of $B$), i.e. all elements in $A$ also belong to $B$
Sets: notions and examples

Example.
Let the universe be the set of all digits

\[ U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]

Let \( A = \{0, 1, 2, 3, 4, 5, 6\} \), \( B = \{2, 3, 5, 7, 9\} \), and \( E = \{0, 2, 4, 6, 8\} \). Then

- \( \overline{A} = U - A = \{7, 8, 9\} \)
- \( A \cup E = \{0, 1, 2, 3, 4, 5, 6, 8\} \)
- \( A \cap E = \{0, 2, 4, 6\} \)
- \( A - B = \{0, 1, 4, 6\} \)
SETS: notions and examples

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- \( \overline{A \cap E} \cap B = \{1, 3, 5, 7, 8, 9\} \cap B = \{3, 5, 7, 9\} \)
- \( \overline{A \cup E} \cap B = \{7, 9\} \cap B = \{7, 9\} \)
- \( A \cap (B \cup E) = A \cap \{0, 2, 3, 4, 5, 6, 7, 8, 9\} = \{0, 2, 3, 4, 5, 6\} \)
- \( (A \cap B) \cup E = \{2, 3, 5\} \cup E = \{0, 2, 3, 4, 5, 6, 8\} \)
SETS: notions and examples

Example.
Let the universe be the set of all digits

\[ U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]

Let \( A = \{0, 1, 2, 3, 4, 5, 6\} \), \( B = \{2, 3, 5, 7, 9\} \), and \( E = \{0, 2, 4, 6, 8\} \). Then

- \( A - (B \cup E) = A - \{0, 2, 3, 4, 5, 6, 7, 8, 9\} = \{1\} \)
- \( (B \cup E) - A = \{0, 2, 3, 4, 5, 6, 7, 8, 9\} - A = \{7, 8, 9\} \)
- \( (B \cap E) - A = \{2\} - A = \emptyset \)
VENN DIAGRAMS

- Shade $A \cap \overline{B}$

$A \cap \overline{B} = A - B$ represents all elements in the universe $U$ that belong to the set $A$, but do not belong to $B$. 
VENN DIAGRAMS

- Shade \((A \cup B) - (A \cap B)\)

\((A \cup B) - (A \cap B)\) represents all elements in the universe \(U\) that belong to \(A\), or \(B\), but do not belong to both \(A\) and \(B\).
VENN DIAGRAMS

- Shade $A \cup \overline{B}$

$$A \cup \overline{B} = (B - A)$$ represents all elements in the universe $U$ that belong to $A$, or do not belong to $B$, or both belong to $A$ and do not belong to $B$. 
VENN DIAGRAMS

- Shade \((A \cap \overline{B}) \cup C\)

Here \((A \cap \overline{B}) \cup C = (A - B) \cup C\) represents all elements in the universe \(U\) that belong to \(C\), or that belong to \(A\) and do not belong to \(B\).
VENN DIAGRAMS

- Shade $A \cap B \cap C$

Here $A \cap B \cap C$ represents all elements in the universe $U$ that belong to $A$ and $B$ and $C$ altogether.
VENN DIAGRAMS

- Shade $A \cap (B \cup C)$

Here $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ represents all elements in the universe $U$ that belong to $A$, and to at least one of the two other sets, $B$ or $B$. 
VENN DIAGRAMS

- Shade \((A \cap B) \cup (A \cap C) \cup (B \cap C)\)

Here \((A \cap B) \cup (A \cap C) \cup (B \cap C)\) represents all elements in the universe \(U\) that belong to at least two of the three sets, \(A\), \(B\) and \(C\).
Rules of set theory:

- Commutative laws:
  \[ E \cup F = F \cup E \quad E \cap F = F \cap E \]

- Associative laws:
  \[ (E \cup F) \cup G = E \cup (F \cup G) \quad (E \cap F) \cap G = E \cap (F \cap G) \]

- Distributive laws:
  \[ (E \cup F) \cap G = (E \cap G) \cup (F \cap G) \]
  \[ (E \cap F) \cup G = (E \cup G) \cap (F \cup G) \]
Notations:

\[ \bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \ldots \cup A_n \]

\[ \bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \ldots \cap A_n \]

Example. \[ \bigcup_{j=1}^{3} A_j = A_1 \cup A_2 \cup A_3 \]

Example. \[ \bigcap_{j=1}^{5} A_j = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \]
De Morgan’s laws:

Consider sets $E_1, E_2, \ldots, E_n$, then

\[
\left( \bigcup_{j=1}^{n} E_j \right) = \bigcap_{j=1}^{n} \overline{E}_j
\]

and

\[
\left( \bigcap_{j=1}^{n} E_j \right) = \bigcup_{j=1}^{n} \overline{E}_j
\]
PROPOSITION CALCULUS

Notiones
We are given two propositions, $p$ and $q$. Then

- $\neg p$ is a proposition “not $p$”
- $p \land q$ is a proposition “$p$ and $q$”
- $p \lor q$ is a proposition “$p$ or $q$” meaning “$p$ or $q$, or both”
- $p \rightarrow q$ (conditional implication) means that the truth of $p$ implies the truth of $q$
There are two values for propositions: \( T \) meaning TRUE and \( F \) meaning FALSE.

If we compare propositions to sets we will observe some similarities. Namely,

\[
\neg p \quad \text{is similar to} \quad \overline{A}
\]

\[
p \land q \quad \text{is similar to} \quad A \cap B
\]

\[
p \lor q \quad \text{is similar to} \quad A \cup B
\]

and \( p \rightarrow q \) is similar to \( A \subseteq B \)

The **truth tables** are similar to Venn diagrams.
TRUTH TABLES

• Given proposition $p$, here is the truth table for $\neg p$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

• Given propositions $p$ and $q$, here is the truth table for $p \lor q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

• Given propositions $p$ and $q$, here is the truth table for $p \land q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>
Introduction to discrete probability.

- Sample space $S = $ the space of all possible outcomes.

- Event = subspace of the sample space $S$.

- Let $E \subseteq S$ be an event. Its complement $\bar{E} = S \setminus E$ is the event “not $E$”.

- $E \cap F = $ event “$E$ and $F$”.

- $E \cup F = $ event “$E$ or $F$”.
Axioms of probability.

- $0 \leq P(E) \leq 1$ for any event $E \subseteq S$

- $P(S) = 1$

- If $E_1 \cap E_2 = \emptyset$ (such sets are said to be disjoint or mutually exclusive), then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
Probability by counting.

- **Example.** A fair die is the one producing each outcome with equal probability. If we roll a fair die, the sample space will consist of six outcomes

  \[ S = \{1, 2, 3, 4, 5, 6\}, \quad \text{with probabilities} \]

  \[ P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6} \]

Let \( E = \{1, 2\} \) be the event of a low score (i.e. 2 or lower). Its probability will be

\[ P(E) = P(1) + P(2) = \frac{2}{6} = \frac{1}{3} \]

Here, since every outcome in the sample space is equally likely,

\[ P(A) = \frac{|A|}{|S|} \quad \text{for each event} \quad A \subseteq S \]
Probability by counting.
In general, in case when every outcome in the sample space $S$ is equally likely,

$$P(A) = \frac{|A|}{|S|}$$

for every event $A \subseteq S$,

where $|A|$ denotes the number of elements in $A$.

• Example. Roll two fair dice. Find the probability of the sum being divisible by three.

Here $S = \begin{bmatrix}
1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\
2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\
3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\
4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\
5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\
6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6
\end{bmatrix}$

Let $E = \{\text{the sum is divisible by three}\}$. Then

$$P(E) = \frac{|E|}{|S|} = \frac{12}{36} = \frac{1}{3}$$
Probability by counting.

- **Example.** Toss a fair coin three times. There each outcome in the sample space

\[ S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \]

is equally likely. We need to find the probability of one head (H) and two tails (T) in the outcome of the three tosses.

Let the event \( E = \{\text{one H and two T}\} \):

\[ S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \]

Since in this example, every outcome in \( S \) is equally likely,

\[ P(E) = \frac{|E|}{|S|} = \frac{\binom{3}{1}}{2^3} = \frac{3}{8} \]
Probability by counting.

• Example. Toss a fair coin $n$ times. There each outcome in the sample space

$$S = \{n\text{-long words of H and T}\}$$

is equally likely. We need to find the probability of $k$ heads (H) and $n - k$ tails (T) in the outcome of the three tosses.

Let the event

$$E = \{n\text{-long words of H and T with } k \text{ Hs and } n - k \text{ Ts}\}$$

Since in this example, every outcome in $S$ is equally likely,

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{n}{k}}{2^n}$$
Probability by counting.

• Example. A deck of 52 cards is dealt among four players. What is the probability that each player gets exactly one ace?

• Solution. Here the cards can be dealt in

\[ |S| = \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!} = \frac{52!}{(13!)^4} \]

different ways, each outcome being equally likely.

Let \( E = \{ \text{each player gets one ace} \} \). Next, we find the number \( |E| \) of the outcomes in \( E \).

We deal the cards in two steps. First, we distribute the four aces among the four players so that each gets exactly one ace. We can do it in \( 4! \) ways. Second, we distribute the rest 48 cards to the four players, each getting 12 cards. This can be done in

\[ \frac{48!}{12! \cdot 12! \cdot 12! \cdot 12!} = \frac{48!}{(12!)^4} \]

ways. Thus, by the multiplicative rule of counting, \( |E| = 4! \cdot \frac{48!}{(12!)^4} \).
Probability by counting.

• **Example.** A deck of 52 cards is dealt among four players. What is the probability that each player gets exactly one ace?

• **Solution (continued).** We have

\[ |S| = \frac{52!}{(13!)^4} \quad \text{and} \quad |E| = 4! \cdot \frac{48!}{(12!)^4} \]

Now, since every outcome in \( S \) is equally likely,

\[ P(E) = \frac{|E|}{|S|} = \frac{4! \cdot 48!}{52!} \cdot (13)^4 \approx 0.1055 \]
Probability by counting.

- **Example.** An urn contains $G$ green and $B$ blue balls. If a random sample of size $n$ is chosen, what is the probability that it contains $k$ green balls?

- **Solution:** We answer the question by counting the number of ways to select $k$ green and $n-k$ blue balls, and then dividing by the total number of possible selections:

$$
\frac{\binom{G}{k} \binom{B}{n-k}}{\binom{G+B}{n}}
$$
Axioms of probability.

- $0 \leq P(E) \leq 1$ for any event $E \subseteq S$

- $P(S) = 1$

- If $E_1 \cap E_2 = \emptyset$ (such sets are said to be disjoint or mutually exclusive), then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
Properties of probability function.

- **Proposition.** $P(\emptyset) = 0$ ($\emptyset$ - the empty set).

- **Proposition.** $P(\overline{E}) = 1 - P(E)$

- **Lemma.** If $E \subseteq F$, then
  
  $$P(F) = P(E) + P(F \cap \overline{E})$$

- **Proposition.** If $E \subseteq F$, then $P(E) \leq P(F)$

- **Inclusion-Exclusion Theorem.**
  
  $$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
Properties of probability function.

- **Proposition.** $P(\emptyset) = 0$ (\emptyset - the empty set).

- **Proof:** The third axiom of probability states:
  
  If $E_1 \cap E_2 = \emptyset$, then
  
  $$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

  Here $S \cap \emptyset = \emptyset$, and therefore
  
  $$P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$$

  Hence $P(\emptyset) = 0$. 
Properties of probability function.

- **Proposition.** \( P(\overline{E}) = 1 - P(E) \)

- **Proof:** The third axiom of probability states:

  If \( E_1 \cap E_2 = \emptyset \), then
  \[
P(E_1 \cup E_2) = P(E_1) + P(E_2)
  \]

  Here \( E \cap \overline{E} = \emptyset \), and therefore
  \[
P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E})
  \]

  where \( P(S) = 1 \) by the second axiom of probability.

  Hence \( P(E) + P(\overline{E}) = 1 \), implying \( P(\overline{E}) = 1 - P(E) \).
Properties of probability function.

- **Proposition.** For any two events $A$ and $B$ in $S$, 
  \[ P(A) = P(A \cap B) + P(A \cap \overline{B}). \]

- **Proof:** Since $A \subseteq S$,
  \[ A = A \cap S = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B}) \]
  by the distributivity law of set theory.

Thus $A = (A \cap B) \cup (A \cap \overline{B})$, where $A \cap B$ is a subset of $B$ and $A \cap \overline{B}$ is a subset of $\overline{B}$.

Therefore $(A \cap B) \cap (A \cap \overline{B}) = \emptyset$, and
\[ P(A) = P((A \cap B) \cup (A \cap \overline{B})) = P(A \cap B) + P(A \cap \overline{B}) \]
by the third axiom of probability.
Properties of probability function.

Lemma. If \( E \subseteq F \), then

\[
P(F) = P(E) + P(F \cap \overline{E})
\]

• Proof: The preceding proposition states that

\[
P(A) = P(A \cap B) + P(A \cap \overline{B})
\]

Now, since \( E \subseteq F \), \( E \cap F = E \) and therefore

\[
P(F) = P(F \cap E) + P(F \cap \overline{E}) = P(E) + P(F \cap \overline{E})
\]
Properties of probability function.

• Proposition. If \( E \subseteq F \), then \( P(E) \leq P(F) \)

• Proof: The preceding lemma states that if \( E \subseteq F \), then

\[
P(F) = P(E) + P(F \cap \overline{E}),
\]

where \( P(F \cap \overline{E}) \geq 0 \) by the first axiom of probability.

Hence

\[
P(F) = P(E) + P(F \cap \overline{E}) \geq P(E) + 0
\]
Inclusion-Exclusion Theorem.

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

• **Proof:** We proved the following lemma:

   If \( E \subseteq F \), then \( P(F) = P(E) + P(F \cap \overline{E}) \).

Taking \( E = A \) and \( F = A \cup B \), we obtain

\[ P(A \cup B) = P(A) + P(B \cap \overline{A}) \]

as \( F \cap \overline{E} = (A \cup B) \cap \overline{A} = (A \cap \overline{A}) \cup (B \cap \overline{A}) = \emptyset \cup (B \cap \overline{A}) = B \cap \overline{A} \)

Also we proved the following proposition:

\[ P(B) = P(B \cap A) + P(B \cap \overline{A}) \]

Hence \( P(B \cap \overline{A}) = P(B) - P(B \cap A) \) and

\[ P(A \cup B) = P(A) + P(B \cap \overline{A}) = P(A) + P(B) - P(A \cap B) \]