

## 1.4 #6

Consider the identity:

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

(a) Is this true? Try it for a few values of  $n$  and  $k$ .

Yes, this is true. Consider  $n = 3$ ,  $k = 3$ :

$$3 \binom{3}{3} = 3(1) = 3, \quad 3 \binom{3-1}{3-1} = 3(1) = 3.$$

Consider  $n = 5$ ,  $k = 3$ :

$$3 \binom{5}{3} = 3 \frac{5!}{2!3!} = 3 \frac{5 \cdot 4}{2} = 30, \quad 5 \binom{5-1}{3-1} = 5 \binom{4}{2} = 5 \frac{4!}{2!2!} = 5 \frac{4 \cdot 3}{2} = 30.$$

Consider  $n = 4$ ,  $k = 2$ :

$$2 \binom{4}{2} = 2 \frac{4!}{2!2!} = 2 \frac{4 \cdot 3}{2} = 12, \quad 4 \binom{4-1}{2-1} = 4 \binom{3}{1} = 4(3) = 12.$$

(b) Use the formula for  $\binom{n}{k}$  to give an algebraic proof of the identity.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

so

$$k \binom{n}{k} = k \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!},$$

and

$$n \binom{n-1}{k-1} = n \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = \frac{n!}{(n-k)!(k-1)!},$$

so we have that

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

□

(c) Give a combinatorial proof of the identity.

Suppose you have  $n$  students, from which you need to select a group of  $k$  students to serve on a committee, one of which needs to be the committee chair.

Option 1: Select the  $k$  students to be on the committee, of which there are  $\binom{n}{k}$  choices, then select one of the  $k$  students in the committee to be the chair, of which there are  $k$  choices. This gives us  $k \binom{n}{k}$  choices.

Option 2: Select one of the  $n$  students to be the committee chair, so we have  $n$  choices, then select  $k-1$  students from the rest of the  $n-1$  students to be the other committee members, which has  $\binom{n-1}{k-1}$  choices. This gives us  $n \binom{n-1}{k-1}$  choices.

Since these both describe the same situation, the two are equivalent.

□

## 1.4 #8

Consider the binomial identity

$$\binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \dots + n \binom{n}{n} = n2^{n-1}.$$

- (a) Give a combinatorial proof of this identity. Hint: What if some number of a group of  $n$  people wanted to go to an escape room, and among those going, one needed to be the team captain?

Say some number of a group of  $n$  people wanted to go to an escape room, and among those going, one needs to be a team captain. The group that wants to go to the escape room could be as small as one person in the group or as large as all members of the group. Say that  $k$  people wanted to go to the escape room. Then there would be  $k$  possible ways to choose a team captain from said group. That is, if of the  $n$  total people, only one wanted to go to an escape room, there would be only one way to select a captain. If two wanted to go to an escape room, then there would be 2 choices for team captain. If three wanted to go, then there are three choices, etc. So we can see that if we want to count the number ways some number of a group of  $n$  people wanted to go to an escape room which required a selection of a team captain, we would have  $\binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \dots + n \binom{n}{n}$  ways to form such a group.

Alternatively, you can select one of the group of  $n$  to be team captain (of which there are  $n$  choices). Then, the remaining  $n - 1$  people have two options each: either they want to go to the escape room or not (giving us another  $2^{n-1}$  possibilities). This gives us  $n2^{n-1}$  ways to form an escape room team from  $n$  people with one team captain.

Since these both describe the same scenario, we have that

$$\binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \dots + n \binom{n}{n} = n2^{n-1}.$$

□

- (b) Give an alternate proof by multiplying out  $(1 + x)^n$  and taking the derivatives of both sides.

From the binomial theorem, we have:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

Taking the derivative with respect to  $x$  of both sides:

$$n(1 + x)^{n-1} = 0 + \binom{n}{1} + 2 \binom{n}{2}x + \dots + n \binom{n}{n}x^{n-1}$$

Then for  $x = 1$ , we have:

$$n(2)^{n-1} = 0 + \binom{n}{1} + 2 \binom{n}{2} + \dots + n \binom{n}{n}.$$

□

## 1.4 #13

Establish the identity below using a combinatorial proof.

$$\binom{2}{2} \binom{n}{2} + \binom{3}{2} \binom{n-1}{2} + \binom{4}{2} \binom{n-2}{2} + \dots + \binom{n}{2} \binom{2}{2} = \binom{n+3}{5}$$

Both sides of this equation represent the number of ways of selecting 5 elements from a group of  $n+3$  elements. It is clear that  $\binom{n+3}{5}$  is choosing 5 from  $n+3$ , so we only need to explain how  $\binom{2}{2} \binom{n}{2} + \binom{3}{2} \binom{n-1}{2} + \binom{4}{2} \binom{n-2}{2} + \dots + \binom{n}{2} \binom{2}{2}$  is choosing 5 from  $n+3$ .

Say you have a deck of cards numbered 1 through  $n+3$  set out in numerical order from least to greatest, and you want to choose 5 of them. The subgroup of 5 that you select can be ordered from least to greatest,  $\{a, b, c, d, e\}$  so that  $a < b < c < d < e$ . Then for the middle card  $c$ , we know that the least  $c$  can be is 3 and the most  $c$  can be is  $n+1$ .

Each term in the sum represents the possible subgroups of 5 for each possible value of  $c$ .

For example, for  $c = 3$ :  $\binom{2}{2} \binom{n}{2}$ . The  $\binom{2}{2}$  represents selecting  $a$  and  $b$  from the two cards lesser than  $c$ , then selecting  $d$  and  $e$  from the last  $n$  cards,  $\binom{n}{2}$ .

For  $c = 4$ , we get the next term in the series: we have 3 options for the 2 cards lesser than  $c$ ,  $\binom{3}{2}$ , and  $n-2$  options for the 2 cards greater than  $c$ ,  $\binom{n-1}{2}$ .

By summing these up for all possible middle card values,  $c = 3$  to  $c = n+1$  we can obtain the total number of ways to select 5 cards from a deck of  $n+3$  cards.

## 1.5 #9

Solve the three counting problems below. Then say why it makes sense that they all have the same answer. That is, say how you can interpret them as each other.

- (a) How many ways are there to distribute 8 cookies to 3 kids?

Say each \* is a cookie. Then we need to divide \*\*\*\*\* into three groups, like \*\*|\*\*|\*\*\*\*.

That is, we have ten total symbols (8 \* and 2 |), and 2 of them must be |, so we have  $\binom{10}{2} = 45$ .

- (b) How many solutions in non-negative integers are there to  $x + y + z = 8$ ?

Again, we can interpret this as a stars and bars problem: we need to have a total of 8 \* divided amongst 3 groups, so we have 8 \* and 2 | for a total of 10 symbols, 2 of which we need to select to be |,  $\binom{10}{2} = 45$ .

- (c) How many different packs of 8 crayons can you make using crayons that come in red, blue, and yellow?

Again, we have 8 crayons that need to be arbitrarily divided into 3 groups (red, blue, and yellow), so 8 \* and 2 |, which gives us  $\binom{10}{2} = 45$ .

If we look at each problem, we can see that they are all a stars and bars problem with 8 stars and 2 bars, so we know that the solution for all of them will be the same:  $\binom{10}{2} = 45$ , since we have a total of 10 symbols (8 stars + 2 bars), and we have to choose 2 of them to be bars.