2.6 #14 (d)

Prove the statement by mathematical induction: The greatest amount of postage you cannot make exactly using 4 and 9 cent stamps is 23 cents.

Proof

First, we should show that it’s not possible to get exactly 23 cents in postage from 4 and 9 cent stamps:

If it were, there would be non-negative integers \( a, b \) such that 23 = 4\( a \) + 9\( b \). Since 9(3) = 27 > 23, we know that \( b \) could only be 0, 1, or 2. We can try all of these:

23 = 4\( a \) + 9(0) = 4\( a \), but 4 does not divide 23, so \( a \) cannot be an integer,
23 = 4\( a \) + 9(1) = 4\( a \) + 9 \(\Rightarrow\) 23 − 9 = 14 = 4\( a \), but again, 4 does not divide 14, so \( a \) cannot be an integer.
And finally, 23 = 4\( a \) + 9(2) = 4\( a \) + 18 \(\Rightarrow\) 23 − 18 = 5 = 4\( a \), and we see that 4 does not divide 5, either, so again, \( a \) cannot be an integer.
That is, there is no way to get exactly 23 cents postage using 4 and 9 cent stamps.

Now we will use induction to show that for all postage greater than 23 cents, you can make exact postage with 4 cent and 9 cent stamps.

We will start with the base case of 24 cents postage:

24 = 4(6) + 9(0).

So we have that for the base case of \( n = 24 \), the condition holds.

Now, for the induction step: So say that for some \( k \in \mathbb{N}, k > 24, \exists a, b \in \mathbb{N} \) such that \( k = 4a + 9b \).

\[ k + 1 = 4a + 9b + 1 = 4(a - 2) + 8 + 9b + 1 = 4(a - 2) + 9(b + 1), \]

or

\[ k + 1 = 4a + 9b + 1 = 4a + 9(b - 3) + 27 + 1 = 4(a + 4) + 9(b - 3). \]

We know that since \( k > 24 \), if \( k = 4a + 9b \), then either \( a \geq 2 \) or \( b \geq 3 \):

If \( a \geq 2 \), then \( a \leq 1 \), as \( a \) must be an integer. If \( a \leq 1 \), then 24 < \( k = 4a + 9b \leq 4 + 9b \ \Rightarrow\ 20 \leq 9b \ \Rightarrow\ \frac{20}{9} \leq b \). Since \( b \) is an integer, this means \( b \geq 3 \).

If \( b \geq 3 \), then \( b \leq 2 \), as \( b \) must be an integer. If \( b \leq 2 \), then 24 < \( k = 4a + 9b \leq 4a + 18 \ \Rightarrow\ 6 \leq 4a \ \Rightarrow\ \frac{6}{4} \leq a \). Since \( a \) is an integer, this means \( a \geq 2 \).

That is, if \( a \geq 2 \), then \( k + 1 = 4(a - 2) + 9(b + 1), \) where \( a - 2, b + 1 \in \mathbb{N} \), and if \( a \geq 2 \), then
\[ k + 1 = 4(a + 4) + 9(b - 3) \] where \( a + 4, b - 3 \in \mathbb{N} \).

So we have that if for some \( k \in \mathbb{N}, k > 23, \exists a, b \in \mathbb{N} : k = 4a + 9b \), then for \( k + 1 \ \exists m, n \in \mathbb{N} : k = 4m + 9n \), and by induction, we have that the greatest amount of postage you cannot make exactly using 4 and 9 cent stamps is 23.

\( \square \)