Topics:

• Recurrence relations.

• Introduction to graphs.

• Examples.

• Paths and cycles.

• Euler cycles.
Recurrence relations.

Complete the sequence satisfying

\[ a_{n+1} = C a_n + D a_{n-1} \]

if the initial terms, \( a_0 \) and \( a_1 \), are known.

That is, find the expression for \( a_n \).
Recurrence relations.

Complete the sequence satisfying

\[ a_{n+1} = a_n + 42a_{n-1} \]

if the initial terms, \( a_0 = 2 \) and \( a_1 = 1 \).
Recurrence relations.

Complete the sequence satisfying

\[ a_{n+1} = a_n + 42a_{n-1} \]

if the initial terms, \( a_0 = 2 \) and \( a_1 = 1 \).

Solving \( x^2 = x + 42 \), we get \( x = 7 \) and \( x = -6 \).

Next, we write \( a_n = a \, 7^n + b \, (-6)^n \) and solve for \( a \) and \( b \). Arriving at

\[ a_n = 7^n + (-6)^n \]
Recurrence relations.

Complete the sequence satisfying

\[ a_{n+1} = 4a_n - 4a_{n-1} \]

if the initial terms, \( a_0 = 5 \) and \( a_1 = 7 \).
Recurrence relations.

Complete the sequence satisfying

\[ a_{n+1} = 4a_n - 4a_{n-1} \]

if the initial terms, \( a_0 = 5 \) and \( a_1 = 7 \).

Solving \( x^2 = 4x - 4 \), we get a **repeated root** \( x = 2 \).

Next, we write \( a_n = a \cdot 2^n + b \cdot n \cdot 2^n \) and solve for \( a \) and \( b \). Arriving at

\[ a_n = 5 \cdot 2^n - 3n \cdot 2^{n-1} \]
Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …
Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Recurrence relation:

\[ a_{n+1} = a_n + a_{n-1} \]

with

\[ a_0 = 1 \text{ and } a_1 = 1 \]
Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Recurrence relation:

\[ a_{n+1} = a_n + a_{n-1} \]

with \( a_0 = 1 \) and \( a_1 = 1 \).

Solving \( x^2 = x + 1 \), we obtain

\[ a_n = a \left( \frac{1 + \sqrt{5}}{2} \right)^n + b \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

Next, we find \( a = \frac{1 + \sqrt{5}}{2\sqrt{5}} \) and \( b = -\frac{1 - \sqrt{5}}{2\sqrt{5}} \).
**Fibonacci numbers.**

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... 

Recurrence relation:

$$a_{n+1} = a_n + a_{n-1}$$

with $a_0 = 1$ and $a_1 = 1$.

Therefore,

$$a_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$
Introduction to graphs.

Graphs consist of vertices, and edges that connect some of them.
Introduction to graphs.

The edges that connect the same pairs of vertices are called **parallel edges**. An edge incident on a single vertex is called a **loop edge**. A vertex not incident on any edge is called **isolated vertex**.
Introduction to graphs.

A graph with neither loops nor parallel edges is called a **simple graph**.
Complete graphs.

The complete graph on \( n \) vertices, denoted by \( K_n \), is the simple graph with \( n \) vertices in which there is an edge between every pair of distinct vertices.

**Question:** Find the number of edges in a complete graph \( K_n \).
Complete graphs.

**Question:** Find the number of edges in a complete graph $K_n$.

**Answer:** Each edge corresponds to a pair of vertices. How many pairs of vertices are there? There are $C(n, 2) = \frac{n(n-1)}{2}$ of them.
Introduction to graphs.

Mathematical notations: graph $G = (V, E)$, where $V$ denotes the set of vertices of the graph, and $E$ the set of all edges in the graph.
A graph $G = (V, E)$ is **bipartite** if there exist subsets $V_1$ and $V_2$ of $V$ such that

$$V_1 \cap V_2 = \emptyset \quad \text{and} \quad V_1 \cup V_2 = V,$$

and each edge in $E$ is incident on one vertex in $V_1$ and one vertex in $V_2$. 
Complete bipartite graphs.

The **complete bipartite graph on** $m$ and $n$ **vertices**, denoted by $K_{m,n}$, is the simple bipartite graph with $m+n$ vertices such that there exist subsets $V_1$ and $V_2$ of $V$ consisting of $m$ and $n$ vertices respectively,

$$V_1 \cap V_2 = \emptyset \quad \text{and} \quad V_1 \cup V_2 = V,$$

with each vertex in $V_1$ connected to each vertex in $V_2$ by an edge in $E$, and $E$ containing no other edges.
Degrees of vertices.

The degree of a vertex $v \in V$ is the number of edges in $E$ incident on $v$. A loop edge counts as two incidences.
A **subgraph** $G'$ of a graph $G = (V, E)$ consists of a subset $V' \subseteq V$ of vertices, and a subset $E' \subseteq E$ of the set of edges in $E$ that are incident to vertices in $V'$ only.
Paths and cycles.

A path from vertex $u \in V$ to $v \in V$ of length $n$ is a sequence of $n + 1$ vertices, alternating with the $n$ edges that connect them, that starts at $u$ and ends at $v$:

$$\{u, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v\}$$

For example: $\{3, e_4, 5, e_6, 6, e_8, 9\}$. In other notation the same path can be written as: $\{3, 5, 6, 9\}$. 
Paths and cycles.

A connected graph is a graph in which for any two vertices in $V$, there is a path connecting them.
Paths and cycles.

A **simple path** from $u$ to $v$ is a path with no repeated vertices. **Example:** $\{3, 6, 9\}$ is a simple path, and $\{3, 6, 7, 9, 6, 5\}$ **is not**.

A **cycle** is a path (of nonzero length) from a vertex $v$ back to itself with no repeated edges. **Example:** $\{3, 6, 5, 3\}$ is a cycle; $\{3, 6, 7, 9, 6, 5, 3\}$ is a cycle; $\{2, e_1, 3, e_2, 2\}$ is a cycle; $\{4, 6, 4\}$ is not a cycle.
A **simple cycle** is a cycle with no repeated vertices.

**Example:** \{3, 6, 5, 3\} is a simple cycle; \{7, 9, 6, 7\} is a simple cycle; \{3, 6, 7, 9, 6, 5, 3\} is a cycle, but **not** a simple cycle.
Königsberg bridge problem.

Königsberg bridge problem.

Lenard Euler (1736): Start at any location in the city. Is it possible to return to the starting location after walking over each bridge exactly once?

A cycle in a graph $G = (V, E)$ that uses each edge in $E$ exactly once is called an Euler cycle.
Euler cycles

The above graph contains an Euler cycle as each of its vertices is of even degree. Here is one:

\{a, b, h, j, f, e, a, c, i, j, g, d, a\}