Topics:

• Pascal’s triangle.

• Generalized combinations.

• Binomial theorem.

• Multinomial theorem.

• Examples.
Combinations.

**Question:** How many $n$-long strings is possible to create with $k$ A’s and $n - k$ B’s?

**Answer:**

$$\binom{n}{k} = \frac{n!}{k!(n - k)!}$$

Another way to denote $\binom{n}{k}$ is $C(n, k)$ (reads “$n$ choose $k$”).
Properties of $\binom{n}{k}$.

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $0 < k < n$
- $\binom{n}{0} = \binom{n}{n} = 1$
Pascal’s triangle.

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{0}{1} &= 1 \\
\binom{2}{0} &= 1 \\
\binom{2}{1} &= ? \\
\binom{3}{0} &= 1 \\
\binom{3}{1} &= ? \\
\binom{4}{0} &= 1 \\
\binom{4}{1} &= ? \\
\binom{4}{2} &= ? \\
\binom{4}{3} &= ? \\
\binom{4}{4} &= 1
\end{align*}
\]
Pascal’s triangle. \[ (\binom{n}{k}) = (\binom{n-1}{k}) + (\binom{n-1}{k-1}) \]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 & \binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 & \binom{2}{1} &= 2 & \binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 & \binom{3}{1} &= ? & \binom{3}{2} &= ? & \binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 & \binom{4}{1} &= ? & \binom{4}{2} &= ? & \binom{4}{3} &= ? & \binom{4}{4} &= 1 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{align*}
\]
Pascal’s triangle. \[ (n \choose k) = (n-1 \choose k) + (n-1 \choose k-1) \]

\[
\begin{align*}
(0) &= 1 \\
1 &= 1 \\
(1) &= 1 \\
(2) &= 1 \\
2 &= 2 \\
(3) &= 1 \\
3 &= 3 \\
(4) &= 1 \\
1 &= ? \\
\vdots & \vdots \\
\end{align*}
\]
Pascal’s triangle.

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 \quad \binom{1}{1} = 1 \\
\binom{2}{0} &= 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1 \\
\binom{3}{0} &= 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1 \\
\binom{4}{0} &= 1 \quad \binom{4}{1} = ? \quad \binom{4}{2} = ? \quad \binom{4}{3} = ? \quad \binom{4}{4} = 1 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]
Pascal’s triangle.

\[
{n \choose k} = \left( {n-1 \choose k} \right) + \left( {n-1 \choose k-1} \right)
\]

\[
\begin{align*}
{n \choose 0} &= 1 \\
{n \choose 1} &= 1
\end{align*}
\]

\[
\begin{align*}
{n \choose 0} &= 1 \\
{n \choose 1} &= 2 \\
{n \choose 2} &= 1
\end{align*}
\]

\[
\begin{align*}
{n \choose 0} &= 1 \\
{n \choose 1} &= 3 \\
{n \choose 2} &= 3 \\
{n \choose 3} &= 1
\end{align*}
\]

\[
\begin{align*}
{n \choose 0} &= 1 \\
{n \choose 1} &= 4 \\
{n \choose 2} &= ? \\
{n \choose 3} &= ? \\
{n \choose 4} &= 1
\end{align*}
\]

\[
\begin{align*}
&\vdots \\
&\vdots \\
&\vdots \\
&\vdots
\end{align*}
\]
Pascal’s triangle.

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1 \\
\binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 \\
\binom{2}{1} &= 2 \\
\binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 \\
\binom{3}{1} &= 3 \\
\binom{3}{2} &= 3 \\
\binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 \\
\binom{4}{1} &= 4 \\
\binom{4}{2} &= 6 \\
\binom{4}{3} &= ? \\
\binom{4}{4} &= 1
\end{align*}
\]
Pascal’s triangle.

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{0}{1} &= 1 \\
\binom{1}{0} &= 1 \\
\binom{1}{1} &= 1 \\
\binom{2}{0} &= 1 \\
\binom{2}{1} &= 2 \\
\binom{2}{2} &= 1 \\
\binom{3}{0} &= 1 \\
\binom{3}{1} &= 3 \\
\binom{3}{2} &= 3 \\
\binom{3}{3} &= 1 \\
\binom{4}{0} &= 1 \\
\binom{4}{1} &= 4 \\
\binom{4}{2} &= 6 \\
\binom{4}{3} &= 4 \\
\binom{4}{4} &= 1 \\
\vdots & \vdots \quad \vdots \quad \vdots
\end{align*}
\]
Generalized combinations.

**Question:** How many \( n \)-long strings is possible to create with 
\( k_1 \) A’s, \( k_2 \) B’s and \( n - k_1 - k_2 \) C’s?

**Answer:**
\[
\binom{n}{k_1 + k_2} \cdot \binom{k_1 + k_2}{k_1} = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!}
\]
Proof: There are two experiments.
(i) Placing \( k = k_1 + k_2 \) ÕEs and \( n - k \) Cs in the \( n \)-long string:
\[
\text{Œ Œ C Œ C C C Œ Œ C C}
\]
This can be done in \( \binom{n}{k} \) ways.

(ii) Replacing ÕEs with \( k_1 \) As and \( k_2 \) Bs:
\[
A \ B \ C \ A \ C \ C \ C \ A \ B \ C \ C
\]
This can be done in \( \binom{k}{k_1} \) ways.

Answer:
\[
\binom{n}{k} \cdot \binom{k}{k_1} = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!}
\]
Generalized combinations.

Example: How many 24-long strings is possible to create with 7 A’s, 11 B’s and 6 C’s?

(e.g. BCBBBCABAACCBBBBAABBABBBACAC)

Answer:

\[
\binom{24}{18} \cdot \binom{18}{7} = \frac{24!}{7! \cdot 11! \cdot 6!}
\]
Generalized combinations.

Question: Count the number of different ways in which $n$ objects of $r$ different types can be ordered if there are

\[
\begin{align*}
&k_1 \text{ objects of type 1,} \\
&k_2 \text{ objects of type 2,} \\
&\ldots \\
&k_r \text{ objects of type } r.
\end{align*}
\]

All in all the total of $k_1 + k_2 + \cdots + k_r = n$ objects.

Answer:

\[
\frac{n!}{k_1! \cdot k_2! \cdots k_r!}
\]
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

- Example.

\[(x + y)^2 = \binom{2}{0} x^0 y^2 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^2 y^0\]

\[= y^2 + 2xy + x^2\]
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

- Example.

\[(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0\]

\[= y^3 + 3xy^2 + 3x^2y + x^3\]
**Binomial Theorem:** \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

- **Proof.** The Binomial theorem follows easily from the definition of \(C(n, k)\).
  Take \(n = 2\),
  \[(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy\]
  Observe that we sum up over all possible 2-long strings that can be created with \(x\) and \(y\),
  \[xx, xy, yx, \text{ and } yy\]
  • There are \(\binom{2}{2} = 1\) strings with two \(x\) and no \(y\)
  • There are \(\binom{2}{1} = 2\) strings with one \(x\) and one \(y\)
  • There are \(\binom{2}{0} = 1\) strings with no \(x\) and two \(y\)

We consolidate the terms to obtaining
\[(x + y)^2 = x^2 + 2xy + y^2\]
Take $n = 3$,

$$(x + y)^3 = (x + y)(x + y)(x + y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

Observe that we sum up over all possible 3-long strings that can be created with $x$ and $y$.

- There are $\binom{3}{3} = 1$ strings with three $x$ and no $y$
- There are $\binom{3}{2} = 3$ strings with two $x$ and one $y$
- There are $\binom{3}{1} = 3$ strings with one $x$ and two $y$
- There are $\binom{3}{0} = 1$ strings with no $x$ and three $y$

We consolidate the terms to obtaining

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
When we factor \((x + y)^n\), each \((x + y)\) contributes either \(x\) or \(y\) to the resulting \(n\)-long string:

\[
(x + y)(x + y)(x + y)\ldots(x + y)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \ldots \quad \downarrow
\]

\[
x \text{ or } y \quad x \text{ or } y \quad x \text{ or } y \ldots x \text{ or } y
\]

Multiplying out \((x + y)(x + y)(x + y)\ldots(x + y)\ldots(x + y)\) we end up with all possible distinct \(n\)-long strings made of \(x\) and \(y\).

We consolidate the terms that have the same number of \(x\) and \(y\):

- There are \(\binom{n}{k}\) strings with \(k\) \(x\) and \((n-k)\) \(y\).

Hence  
\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]
Binomial Theorem: \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

This is why the numbers \(\binom{n}{k}\) are also known as binomial coefficients.

- **Example.** Find the coefficient in front of \(x^7y^9\) in the expansion of \((x+y)^{16}\).

- **Answer:** \(\binom{16}{7} = 11,440\)
Binomial Theorem: \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

- **Example.** Find \(\sum_{k=0}^{n} \binom{n}{k}\).

- **Solution:** Observe that if we use the Binomial Theorem with \(x = y = 1\), the left hand side \(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\) becomes \(\sum_{k=0}^{n} \binom{n}{k}\).

Thus

\[
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1 + 1)^n = 2^n
\]
Generalized combinations.

**Question:** Count the number of different ways in which \( n \) objects of \( r \) different types can be ordered if there are

\[
\begin{align*}
    k_1 \text{ objects of type 1,} \\
    k_2 \text{ objects of type 2,} \\
    \ldots \\
    k_r \text{ objects of type } r.
\end{align*}
\]

All in all the total of \( k_1 + k_2 + \cdots + k_r = n \) objects.

**Answer:**

\[
\frac{n!}{k_1! \cdot k_2! \cdots \cdot k_r!}
\]
**Multinomial Theorem**

\[(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1 \geq 0, \ldots, k_r \geq 0 \atop k_1 + k_2 + \cdots + k_r = n} \frac{n!}{k_1! \cdot k_2! \cdots k_r!} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}\]

**Example:**

\[(x + y + z)^2 = \sum_{0 \leq k_1, k_2, k_3 \leq 2 \atop k_1 + k_2 + k_3 = 2} \frac{2!}{k_1! k_2! k_3!} x^{k_1} y^{k_2} z^{k_3}\]

\[= \frac{2!}{2!0!0!} x^2 y^0 z^0 + \frac{2!}{0!2!0!} x^0 y^2 z^0 + \frac{2!}{0!0!2!} x^0 y^0 z^2 + \frac{2!}{0!1!1!} x^0 y^1 z^1 + \frac{2!}{1!0!1!} x^1 y^0 z^1 + \frac{2!}{1!1!0!} x^1 y^1 z^0\]

\[= x^2 + y^2 + z^2 + 2yz + 2xz + 2xy\]

Multinomial Theorem can be proved in the same way as the Binomial Theorem. Alternatively, it can be proved using the Binomial Theorem.
Question: How many strings can be formed with the following letters?

\[ S \ H \ E \ S \ E \ L \ L \ S \ S \ E \ A \ S \ H \ E \ L \ L \ S \]

Solution: There are

\[
\begin{align*}
  k_1 &= 1 \text{ letter } A, \\
  k_2 &= 4 \text{ letters } E, \\
  k_3 &= 2 \text{ letters } H, \\
  k_4 &= 4 \text{ letters } L, \\
  k_5 &= 6 \text{ letters } S \\
\end{align*}
\]

Therefore there must be

\[
\frac{17!}{1!4!2!4!6!}
\]

different strings made with these letters.
**Question:** In how many ways can 52 different cards be dealt between four players (call them Player 1, Player 2, Player 3, and Player 4) where each gets exactly 13 cards?

**Solution:** Line up the cards: 1, 2, ..., 52

Each card is assigned a player. Thus there will be 4 types of cards: the ones that belong to Player 1, the ones that belong to Player 2, the ones that belong to Player 3, and the ones that belong to Player 4. Hence there are \( \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!} \) ways to deal the deck.