MTH 355 - Lecture 11

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Topics:

- Mathematical induction.
MATHEMATICAL INDUCTION
Let the domain be all positive integers. We need to prove $P(n)$ is true for all $n$ in the domain.

Mathematical Induction:
• BASIS STEP: Verify $P(1)$ is true.

• INDUCTIVE STEP: Show the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all $k$ in the domain.

\[
\left( P(1) \land \forall k \left( P(k) \rightarrow P(k + 1) \right) \right) \rightarrow \forall n P(n)
\]
MATHEMATICAL INDUCTION

Example: Prove that
\[ 1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2} \]
for all positive integers \( n \).

Proof:
BASIS: \( P(1) \) is true as \( 1 = \frac{1 \cdot (1+1)}{2} \).

INDUCTIVE STEP: Here
\[ P(k) : \quad 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2} \]
\[ P(k+1) : \quad 1+2+3+\ldots+k+(k+1) = \frac{(k+1)(k + 2)}{2} \]
MATHEMATICAL INDUCTION

INDUCTIVE STEP: Here

\( P(k) : \quad 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \)

\( P(k+1) : \quad 1+2+3+\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2} \)

Suppose \( P(k) \) is true, then

\[
1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \\
= \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{k + 1}{2} \cdot (k + 2) = \frac{(k + 1)(k + 2)}{2}
\]

Thus, \( P(k) \) implies \( P(k+1) \).

Q.E.D.
MATHEMATICAL INDUCTION

Example: Prove that $n^2 > n + 1$ for all integer $n \geq 2$.

Proof:
Basis: True when $n = 2$ as $2^2 > 2 + 1$.

Inductive Step: Suppose the statement is valid for some $k \geq 2$. We need to show it is valid for $k + 1$. That is, given $k^2 > k + 1$, we need to prove 

$$(k + 1)^2 > (k + 1) + 1 = k + 2$$

Here is how we do it: $k^2 > k + 1$ implies 

$$(k+1)^2 = k^2 + 2k + 1 > (k+1) + 2k + 1 = 3k + 2 > k + 2$$

Q.E.D.
MATHEMATICAL INDUCTION

Example: Prove that

\[ 4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1) \]

for all positive integers \( n \).

Proof:

BASIS: True when \( n = 1 \) as \( 4 = 1 \cdot (3 + 1) \).

INDUCTIVE STEP: Suppose the statement is valid for some integer \( k > 0 \). We need to show it is valid for \( k + 1 \). That is, given

\[ 4 + 10 + 16 + \cdots + (6k - 2) = k(3k + 1), \]

we need to prove

\[ 4 + \cdots + (6k - 2) + (6(k+1) - 2) = (k+1)(3(k+1) + 1) \]
So, we need to prove
\[ 4 + \cdots + (6k - 2) + (6(k + 1) - 2) = (k + 1)(3(k + 1) + 1) \]
We do it as follows.

\[ 4 + \cdots + (6k - 2) = k(3k + 1) \]
implies
\[ 4 + \cdots + (6k - 2) + (6(k + 1) - 2) = k(3k + 1) + (6(k + 1) - 2) \]
\[ = 3k^2 + 7k + 4 \]
\[ = (k + 1)(3k + 4) \]
\[ = (k + 1)(3(k + 1) + 1) \]

Q.E.D.
MATHEMATICAL INDUCTION

Example: Prove that

\[ 1 + 7 + 13 + \cdots + (6n - 5) = n(3n - 2) \]

for all integer \( n \geq 1 \).

Proof:
Basis: True when \( n = 1 \) as there

\[ 1 + 7 + 13 + \cdots + (6n - 5) = 1 = n(3n - 2) \]

Inductive Step: Suppose the statement is valid for some integer \( k > 0 \). We need to show it is valid for \( k + 1 \). That is, given

\[ 1 + 7 + 13 + \cdots + (6k - 5) = k(3k - 2), \]

we need to prove

\[ 1 + 7 + \cdots + (6k - 5) + (6(k+1) - 5) = (k+1)(3(k+1) - 2) \]
MATHEMATICAL INDUCTION

So, we need to prove

\[ 1 + 7 + \cdots + (6k-5) + (6(k+1)-5) = (k+1)(3(k+1)-2) \]

We do it as follows.

\[ 1 + 7 + 13 + \cdots + (6k-5) = k(3k - 2) \]
implies

\[ 1 + 7 + \cdots + (6k-5) + (6(k+1)-5) = k(3k - 2) + (6(k+1)-5) \]
\[ = 3k^2 + 4k + 1 \]
\[ = (k + 1)(3k + 1) \]
\[ = (k + 1)(3(k + 1) - 2) \]

Q.E.D.