MTH 355 - Lecture 6

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Topics:

• Methods of Proof
Methods of Proof: Direct Proofs

Given a set of hypotheses $p_1, p_2, \ldots, p_n$ formulated as $n$ statements and a conclusion statement $q$.

A **direct proof** showing a conditional statement

$$(p_1 \land p_2 \land \ldots \land p_n) \to q$$

is true is done by validating a sequence of inferences

$$(p_1 \land p_2 \land \ldots \land p_n) \to r_1$$

$$r_1 \to r_2$$

$$\vdots$$

$$r_{m-1} \to r_m$$

$$r_m \to q$$
Methods of Proof: by Contraposition

Given a set of hypotheses \( p_1, p_2, \ldots, p_n \) formulated as \( n \) statements and a conclusion statement \( q \).

A **proof by contraposition** shows a conditional statement

\[
(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q
\]

is **true** by validating a logically equivalent conditional statement

\[
\neg q \rightarrow \neg (p_1 \land p_2 \land \cdots \land p_n)
\]
Methods of Proof: by Contradiction

Suppose we want to prove some statement $q$ is true.

A proof by contradiction works by constructing a statement $r$ for which it can be shown that

$$
\neg q \rightarrow (r \land \neg r)
$$

Now, since $r \land \neg r$ is always false, $\neg q$ must be false.

Hence, we prove that $q$ is true.
Methods of Proofs: by Contradiction

**Example:** Prove that $\sqrt{2}$ is irrational.

**Proof:** Suppose not. Suppose $\sqrt{2}$ is rational. That means it can be represented as

$$\sqrt{2} = \frac{m}{n},$$

where $n$ and $m$ are integers, and $m > 0$.

Since we can reduce fraction if necessary, we can assume that $n$ and $m$ have no common factors. Thus they cannot be both even.
Methods of Proof: by Contradiction

Then

\[ 2 = (\sqrt{2})^2 = \frac{m^2}{n^2} \]

and \( m^2 = 2n^2 \). Thus \( m^2 \) is even. This implies \( m \) is even.

So \( m = 2k \), where \( k \) is some integer. Then

\[ n^2 = \frac{m^2}{2} = \frac{4k^2}{2} = 2k^2 \]

is an even number. Hence \( n \) is even. But then \( n \) and \( m \) are both even, contradicting our earlier statement.

Therefore \( \sqrt{2} \) is not a rational number.

Q.E.D.
Methods of Proof: by cases

A proof by cases must cover all possible cases that arise in a theorem.

Suppose we need to prove a statement that can be expressed in the form

$$(p_1 \lor p_2 \lor \ldots \lor p_n) \rightarrow q,$$

where the hypothesis is made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$.

We observe that

$$\left( (p_1 \lor p_2 \lor \ldots \lor p_n) \rightarrow q \right) \equiv \left( (p_1 \rightarrow q) \land (p_2 \rightarrow q) \land \ldots \land (p_n \rightarrow q) \right)$$

Thus it can be proven by proving each conditional statement $p_i \rightarrow q$. 
Methods of Proof: by cases

Example: Prove that $|xy| = |x| \cdot |y|$ for any real $x$ and $y$.

Proof: There are four cases,
Case I: $x \geq 0$ and $y \geq 0$. There

$$|x| \cdot |y| = xy = |xy| \quad \text{as } xy \geq 0$$

Case II: $x \geq 0$ and $y < 0$. There

$$|x| \cdot |y| = x \cdot (-y) = -xy = |xy| \quad \text{as } xy \leq 0$$

Case III: $x < 0$ and $y \geq 0$. There

$$|x| \cdot |y| = (-x) \cdot y = -xy = |xy| \quad \text{as } xy \leq 0$$

Case IV: $x < 0$ and $y < 0$. There

$$|x| \cdot |y| = (-x) \cdot (-y) = xy = |xy| \quad \text{as } xy > 0$$

Q.E.D.
Methods of Proof: by cases

Example: Prove that for any real $x$ and $y$, their minimum

$$\min\{x, y\} = \frac{x + y - |x - y|}{2}$$

Proof: Consider the following two cases.

• Case I: $x \geq y$

There $|x - y| = x - y \geq 0$ and $\min\{x, y\} = y$.

Hence

$$\min\{x, y\} = y = \frac{x + y - (x - y)}{2} = \frac{x + y - |x - y|}{2}$$
Methods of Proof: by cases

Now consider

• Case II: \( x < y \)

There \( |x - y| = y - x > 0 \) and \( \min\{x, y\} = x \).

Hence

\[
\min\{x, y\} = x = \frac{x + y - (y - x)}{2} = \frac{x + y - |x - y|}{2}
\]

So in all possible cases, \( x \geq y \) and \( x < y \), the statement is valid.
Prove or Disprove

**Example.** Prove or disprove the following statement:

*A product of any two rational numbers is rational.*
Prove or Disprove

Example. Prove or disprove the following statement:

A product of any two irrational numbers is irrational.
Prove or Disprove

Example. Prove or disprove the following statement:

If $d$ is a positive integer, then $\sqrt{d}$ is irrational.
Prove or Disprove

Example. Prove or disprove the following statement:

If $d$ is a positive integer that is not a perfect square, then $\sqrt{d}$ is irrational.