MTH 312

Lectures 15-17

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- Topics: Lebesgue measure on $\mathbb{R}$.
- Topics: properties of Lebesgue measure on $\mathbb{R}$.
- Lebesgue's criterion for Riemann integrability.
- Topics: the Lebesgue integral.

Riemann integrability

Recall that for $c \in [a,b]$, the function $f_c(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$ is Riemann integrable on $[a,b]$, and

$$\int_a^b f_c(x)\,dx = 0.$$

Example. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$. Since the set $\mathbb{Q}$ and its complement are everywhere dense sets in $\mathbb{R}$, for any closed interval $[a,b]$ and any partition P, $L(f,P) = 0$ and $U(f,P) = b - a$. Thus,

$L(f) = 0$ and $U(f) = b - a$, and $f(x)$ is not Riemann integrable.

On the other hand, if we enumerate $\mathbb{Q} \cap [a,b] = \{c_1, c_2, c_3, \ldots\}$, then

$$f(x) = \sum_{k=1}^{\infty} f_{c_k}(x) \quad \text{and} \quad \sum_{k=1}^{\infty} \int_a^b f_{c_k}(x)\,dx = 0.$$

Lebesgue measure on $\mathbb{R}$.

Definition. For $A \subseteq \mathbb{R}$, define its Lebesgue measure by letting

$$m(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \ : \ A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}.$$

It is also called Lebesgue outer measure.

Proposition. If A is a singleton (i.e., $A = \{a\}$), then $m(A) = 0$.

Proposition. If $A \subseteq B$, then $m(A) \leq m(B)$.

Proposition. $m\big([a, b]\big) = b - a$.

Proof: By Heine–Borel theorem, every cover $[a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$ has a finite subcover $[a, b] \subseteq \bigcup_{i=1}^{n} (a_i, b_i)$. Moreover, we can reduce and enumerate the collection of intervals so that $a_i < b_{i-1} < b_i$. Then,

$$\sum_{i=1}^{\infty} (b_i - a_i) \geq \sum_{i=1}^{n} (b_i - a_i) > b_n - a_1 > b - a.$$

Lebesgue measure on $\mathbb{R}$.

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It is also called Lebesgue outer measure.

Proposition. If A is a singleton (i.e., $A = \{a\}$), then $m(A) = 0$.

Proposition. If the set $A = \{a_1, a_2, \ldots\}$ consists of countably many or finitely many points, then $m(A) = 0$.

Proof: Suppose $A = \{a_1, a_2, \ldots\}$ is countably infinite. For a given $\epsilon > 0$,

$$A \subseteq \bigcup_{i=1}^{\infty} \left(a_i - \frac{\epsilon}{2^{i+1}}, \, a_i + \frac{\epsilon}{2^{i+1}} \right) \quad \text{with} \quad \sum_{i=1}^{\infty} \left(\left[a_i + \frac{\epsilon}{2^{i+1}} \right] - \left[a_i - \frac{\epsilon}{2^{i+1}} \right] \right) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Hence, since $\epsilon > 0$ was arbitrary, we have $m(A) = 0$. The same approach works for finitely many points.

Lebesgue measure on $\mathbb{R}$.

Definition. A set $E \subseteq \mathbb{R}$ is Lebesgue measurable (or simply measurable) if for any $A \subseteq \mathbb{R}$,

$$m(A) = m\big(A \cap E\big) + m\big(A \cap \overline{E}\big),$$

i.e., it satisfies Carathéodory criterion.

Definition. A collection of sets in $\mathbb{R}$ is a σ-algebra if it is closed under countable unions, countable intersections, and taking complements.

Proposition. Let Σ be a σ-algebra. Then,

- For $A_1, A_2, \ldots, A_n \in \Sigma$, we have $\bigcap\limits_{i=1}^{n} A_i \in \Sigma$ and $\bigcup\limits_{i=1}^{n} A_i \in \Sigma$;
- $\emptyset, \mathbb{R} \in \Sigma$.

Proof: Countable intersection $A_1 \cap \ldots \cap A_{n-1} \cap A_n \cap A_n \cap A_n \cap \ldots \in \Sigma$, and countable union $A_1 \cup \ldots \cup A_{n-1} \cup A_n \cup A_n \cup A_n \cup \ldots \in \Sigma$. For any $A \in \Sigma$, $\emptyset = A \cap \overline{A} \in \Sigma$. Finally, $\mathbb{R} = \overline{\emptyset} \in \Sigma$.

Lebesgue measure on $\mathbb{R}$.

Theorem. Let $\mathcal{F}$ denote the space of all measurable sets in $\mathbb{R}$. Then, $\mathcal{F}$ is a σ-algebra.

Proof: If $E \in \mathcal{F}$, then $m(A) = m\big(A \cap E\big) + m\big(A \cap \overline{E}\big)$ implies $\overline{E} \in \mathcal{F}$. Next, for $E, F \in \mathcal{F}$, and any $A \subseteq \mathbb{R}$,

$$m(A) = m\big(A\cap E\big)+m\big(A\cap\overline{E}\big) = m\big(A\cap E\cap F\big)+m\big(A\cap E\cap\overline{F}\big)+m\big(A\cap\overline{E}\cap F\big)+m\big(A\cap\overline{E}\cap\overline{F}\big)$$

$$= m\big(A \cap E \cap F\big) + m\big(A \cap \overline{E \cap F}\big)$$

as

$$m\big(A \cap \overline{E \cap F}\big) = m\big(A\cap(\overline{E}\cup\overline{F})\big) = m\big(A\cap(\overline{E}\cup\overline{F})\cap E\big)+m\big(A\cap(\overline{E}\cup\overline{F})\cap\overline{E}\big)$$

$$= m\big(A\cap E\cap\overline{F}\big)+m\big(A\cap\overline{E}\big) = m\big(A\cap E\cap\overline{F}\big)+m\big(A\cap\overline{E}\cap F\big)+m\big(A\cap\overline{E}\cap\overline{F}\big).$$

Thus, $E \cap F \in \mathcal{F}$. De Morgan's laws yield $E \cup F = \overline{(\overline{E} \cap \overline{F})} \in \mathcal{F}$.

Lebesgue measure on $\mathbb{R}$.

Theorem. Let $\mathcal{F}$ denote the space of all measurable sets in $\mathbb{R}$. Then, $\mathcal{F}$ is a σ-algebra.

Definition. Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing all open (and therefore, all closed) sets. The elements of $\mathcal{B}$ are called Borel sets.

Theorem. Borel sets are measurable, i.e., $\mathcal{B} \subset \mathcal{F}$.

Proof: For any $a \in \mathbb{R}$, $m(A) = m\big(A \cap E\big) + m\big(A \cap \overline{E}\big)$ is established for $A = (a, \infty)$. For all the intervals (a_i, b_i) in the covering of A, if $a \in (a_i, b_i)$, we substitute (a_i, b_i) with (a_i, a) and $(a - \epsilon/2^i, b_i)$.

Proposition. If $m(A) = 0$, then A is measurable, i.e., $A \in \mathcal{F}$.

Lebesgue's criterion for Riemann integrability.

Lebesgue's Theorem. Consider a bounded function $f(x)$ on a closed interval $[a,b]$. Function $f(x)$ is Riemann integrable on $[a,b]$ if and only if the set of all points of discontinuity has Lebesgue measure zero.

Example. Let $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$. Since the set $\mathbb{Q}$ and its complement are everywhere dense sets in $\mathbb{R}$, for any closed interval $[a,b]$ and any partition P, $L(f,P)=0$ and $U(f,P)=b-a$. Thus,

$L(f)=0$ and $U(f)=b-a$, and $f(x)$ is not Riemann integrable.

Indeed, $f(x)$ is discontinuous on all of $[a,b]$ (i.e., the set of all points of discontinuity has measure $b-a$), and by Lebesgue's Theorem, $f(x)$ is not Riemann integrable.

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

For a set $E \subseteq \mathbb{R}$ let

$$f_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in \overline{E} \end{cases}.$$

We want

$$\int_A f_E(x)\,dx = m(A \cap E).$$

Consider $A \subseteq \mathbb{R}$ satisfying $m(A) < \infty$. Now, $f_E(x) + f_{\overline{E}}(x) = 1$ and

$$\int_A f_E(x)\,dx = m(A \cap E), \qquad \int_A f_{\overline{E}}(x)\,dx = m(A \cap \overline{E}), \quad \text{and} \quad \int_A dx = m(A).$$

Thus, we need

$$m(A) = m\big(A \cap E\big) + m\big(A \cap \overline{E}\big),$$

i.e., $E \in \mathcal{F}$.

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

Consider a measure space $(\mathbb{R}, \mathcal{F}, m)$, where $\mathbb{R}$ is the space, $\mathcal{F}$ is the σ-algebra of all measurable sets, and $m : \mathcal{F} \to [0, \infty]$ is the Lebesgue outer measure.

Definition. A function f is said to be measurable if

$$f^{-1}(B) = \{x : f(x) \in B\} \in \mathcal{F} \quad \text{for every Borel set } B \in \mathcal{B}.$$

Notation: $f \in \mathcal{F}$.

Example. If E is measurable (i.e., $E \in \mathcal{F}$), then $f_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in \overline{E} \end{cases}$ is measurable since for every $B \in \mathcal{B}$, set $f_E^{-1}(B)$ equals either $\emptyset$, $\mathbb{R}$, E, or $\overline{E}$.

Almost sure equivalence: measurable functions f and g are equal almost everywhere (a.e.) if

$$m\Big(\big\{x \in \mathbb{R} : f(x) \neq g(x)\big\}\Big) = 0.$$

Notation: $f(x) = g(x)$ **a.e.**

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

Consider a measure space $(\mathbb{R}, \mathcal{F}, m)$, where $\mathbb{R}$ is the space, $\mathcal{F}$ is the σ-algebra of all measurable sets, and $m : \mathcal{F} \to [0, \infty]$ is the Lebesgue outer measure.

Partition y-axis into subintervals of size $\leq \delta$, i.e., $0 < y_{i+1} - y_i \leq \delta$.

Recall: for a measurable function f over a set $A \in \mathcal{F}$,

$$f^{-1}[y_i, y_{i+1}) = \Big\{ x \in A \,:\, f(x) \in [y_i, y_{i+1}) \Big\}.$$

The **Lebesgue integral** of a measurable function f over a set $A \in \mathcal{F}$

$$\int_A f(x)\, dx, \qquad \text{also denoted by} \qquad \int_A f(x)\, dm(x),$$

is defined as the limit of

$$\sum_i y_i^* \, m\Big(f^{-1}[y_i, y_{i+1})\Big), \qquad y_i^* \in [y_i, y_{i+1}],$$

as $\delta \to 0+$.

Example. For $E \in \mathcal{F}$, $f_E(x)$ is measurable, and $\int_A f_E(\mathbb{R})\, dx = m(A \cap E)$.

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

Consider a measure space $(\mathbb{R}, \mathcal{F}, m)$, where $\mathbb{R}$ is the space, $\mathcal{F}$ is the σ-algebra of all measurable sets, and $m : \mathcal{F} \to [0, \infty]$ is the Lebesgue outer measure.

Lebesgue integral

$$\int\limits_A f(x)\, dx$$

of a measurable function f over a set $A \in \mathcal{F}$ is defined as the limit as $\delta \to 0+$ of

$$\sum_i y_i^*\, m\Big(f^{-1}[y_i, y_{i+1})\Big), \qquad y_i^* \in [y_i, y_{i+1}].$$

Properties:

- If $f(x) = g(x)$ **a.e.** then $\int\limits_A f(x)\, dx = \int\limits_A g(x)\, dx$ for all $A \in \mathcal{F}$.
- If $m(A) = 0$, then $\int\limits_A f(x)\, dx = 0$ for all $f \in \mathcal{F}$.

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

Consider a measure space $(\mathbb{R}, \mathcal{F}, m)$, where $\mathbb{R}$ is the space, $\mathcal{F}$ is the σ-algebra of all measurable sets, and $m : \mathcal{F} \to [0, \infty]$ is the Lebesgue outer measure.

Monotone Convergence Theorem. Consider a set $A \in \mathcal{F}$ and a sequence $f_n : \mathbb{R} \to [0, \infty]$ of nonnegative measurable functions satisfying

$$0 \leq f_n(x) \leq f_{n+1}(x)$$

for all $n \in \mathbb{N}$ and all $x \in A$. Then,

$$\lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A \lim_{n\to\infty} f_n(x)\, dx.$$

Fatou's Lemma. Consider a set $A \in \mathcal{F}$ and a sequence $f_n : \mathbb{R} \to [0, \infty]$ of nonnegative measurable functions. Then,

$$\int_A \liminf_{n\to\infty} f_n(x)\, dx \leq \liminf_{n\to\infty} \int_A f_n(x)\, dx.$$

Lebesgue measure and Lebesgue integral on $\mathbb{R}$.

Consider a measure space $(\mathbb{R}, \mathcal{F}, m)$, where $\mathbb{R}$ is the space, $\mathcal{F}$ is the σ-algebra of all measurable sets, and $m : \mathcal{F} \to [0, \infty]$ is the Lebesgue outer measure. A measurable function $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable if

$$\int_{-\infty}^{\infty} |g(x)|\, dx < \infty.$$

Lesbegue's Dominated Convergence Theorem. Consider a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of measurable functions satisfying

$$f_n(x) \to f(x) \qquad \text{pointwise on } \mathbb{R}$$

and dominated by an Lebesgue integrable function $g : \mathbb{R} \to [0, \infty)$, i.e.,

$$|f_n(x)| \leq g(x) \qquad \text{a.e. } \forall n.$$

Then,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$