MTH 312

Lectures 10-14

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- The Riemann integral.
- Riemann integrability.
- Integrating functions with discontinuities.
- Properties of the Riemann integral.
- The Fundamental Theorem of Calculus.
- Mean Value Theorem for Definite Integrals.

The Riemann Integral.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$, i.e., $\exists M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Definition. A finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ is said to be a partition of $[a, b]$ if

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

In other words, $[a, b]$ is partitioned into subintervals $[x_{k-1}, x_k]$.

Let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$.

Definition. The lower sum of $f(x)$ with respect to partition P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

The upper sum of $f(x)$ with respect to partition P is given by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Notice that $L(f, P) \leq U(f, P)$.

The Riemann Integral.

Consider a bounded function $f(x)$ on a closed interval $[a,b]$, i.e., $\exists M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Definition. Consider partitions P and P' of $[a,b]$. Partition P' is said to be a refinement of partition P if $P \subseteq P'$.

Lemma. If partition P' is a refinement of partition P, then

$$L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P).$$

Proof: First, the above inequality is established for the case when P' is obtained from P by adding a single point, i.e., $P' = P \cup \{x'\}$. The general statement follows by iteratively adding single points to P, obtaining P'.

Definition. Let $\mathcal{P}$ be the set of all possible partitions of $[a,b]$. Then, the lower integral of $f(x)$ is defined by

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}.$$

The upper integral of $f(x)$ is defined by

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}.$$

Notice that $L(f,P) \leq L(f)$ and $U(f) \leq U(f,P)$ for all $P \in \mathcal{P}$.

The Riemann Integral.

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The upper integral of $f(x)$ is defined by

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}.$$

Proposition. $L(f) \le U(f)$.

Proof: Suppose $d = L(f) - U(f) > 0$. Then, there are partitions P' and P'' such that

$$L(f) - d/3 \le L(f,P') \le L(f) \quad \text{and} \quad U(f) \le U(f,P'') \le U(f) + d/3.$$

Partition $P = P' \cup P''$ is a refinement to both P' and P'', and

$$d = L(f) - U(f) \le 2d/3 + L(f,P') - U(f,P'') \le 2d/3 + L(f,P) - U(f,P) \le 2d/3$$

arriving at contradiction.

The Riemann Integral.

Consider a bounded function $f(x)$ on a closed interval $[a,b]$.

Definition. Let $\mathcal{P}$ be the set of all possible partitions of $[a,b]$. Then, the lower integral of $f(x)$ is defined by

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}.$$

The upper integral of $f(x)$ is defined by

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}.$$

Proposition. $L(f) \le U(f)$.

Definition. A bounded function $f(x)$ on $[a,b]$ is Riemann integrable (or just integrable) if $L(f) = U(f)$, in which case this quantity is called the Riemann integral, and denoted by $\int_a^b f(x)\,dx$, i.e.,

$$\int_a^b f(x)\,dx = L(f) = U(f).$$

Riemann integrality.

Consider a bounded function $f(x)$ on a closed interval $[a,b]$.

Theorem. Bounded function $f(x)$ is Riemann integrable if and only if for any $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$0 \le U(f,P) - L(f,P) \le \epsilon.$$

Proof: Suppose for any $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $0 \le U(f,P) - L(f,P) \le \epsilon$. Since $L(f,P) \le L(f) \le U(f) \le U(f,P)$,

$$0 \le U(f) - L(f) \le U(f,P) - L(f,P) \le \epsilon.$$

Hence, $U(f) - L(f) \le \epsilon$ for all $\epsilon > 0$. Therefore, $U(f) = L(f)$.

Conversely, suppose $U(f) = L(f)$. Then, there exist partitions P' and P'' such that

$$L(f) - \epsilon/2 \le L(f,P') \le L(f) = U(f) \le U(f,P'') \le U(f) + \epsilon/2.$$

Hence, for $P = P' \cup P''$, a refinement to both P' and P'',

$$0 \le U(f,P) - L(f,P) \le U(f,P'') - L(f,P') \le U(f) - L(f) + \epsilon = \epsilon.$$

Theorem. A continuous function $f(x)$ on $[a,b]$ is Riemann integrable.

Riemann integrality.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Theorem. Bounded function $f(x)$ is Riemann integrable if and only if for any $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$0 \leq U(f, P) - L(f, P) \leq \epsilon.$$

Theorem. A continuous function $f(x)$ on $[a, b]$ is Riemann integrable.

Proof: A continuous function on $[a, b]$ is uniformly continuous.

Thus, $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $|x - y| < \delta \; \Rightarrow \; |f(x) - f(y)| < \dfrac{\epsilon}{b - a}$.

Thus, for a partition P with $0 < \triangle x_k = x_k - x_{k-1} < \delta$ for all k, $M_k - m_k < \frac{\epsilon}{b-a}$, and therefore

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \leq \frac{\epsilon}{b - a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \epsilon.$$

Properties of the Riemann integral.

Consider a bounded function $f(x)$ on a closed interval $[a,b]$.

Theorem. Fix $c \in (a,b)$. A bounded function $f(x)$ is Riemann integrable on $[a,b]$ if and only if $f(x)$ is Riemann integrable on $[a,c]$ and $[c,b]$. In the later case,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof: For any partition P of $[a,b]$, partition $P_c = P \cup \{c\}$ is a refinement, and

$$L(f,P) \le L(f,P_c) \le U(f,P_c) \le U(f,P).$$

Therefore, letting $\mathcal{P}_{[\alpha,\beta]}$ denote the set of all possible partitions of $[\alpha,\beta]$, we have the left and the right integrals over $[a,b]$ represented as

$$L(f) = \sup\{L(f,P_c) : P \in \mathcal{P}_{[a,b]}\} \quad \text{and} \quad U(f) = \inf\{U(f,P_c) : P \in \mathcal{P}_{[a,b]}\}.$$

The statement of the theorem follows from

$$\{P_c : P \in \mathcal{P}_{[a,b]}\} = \{P' \cup P'' : P' \in \mathcal{P}_{[a,c]},\, P'' \in \mathcal{P}_{[c,b]}\}.$$

Properties of the Riemann integral.

Proof (cont.): For any partition P of $[a,b]$, partition $P_c = P \cup \{c\}$ is a refinement, and

$$L(f,P) \le L(f,P_c) \le U(f,P_c) \le U(f,P).$$

Therefore, letting $\mathcal{P}_{[\alpha,\beta]}$ denote the set of all possible partitions of $[\alpha,\beta]$, we have the left and the right integrals over $[a,b]$ represented as

$$L(f) = \sup\{L(f,P_c) : P \in \mathcal{P}_{[a,b]}\} \quad \text{and} \quad U(f) = \inf\{U(f,P_c) : P \in \mathcal{P}_{[a,b]}\}.$$

The statement of the theorem follows from

$$\{P_c : P \in \mathcal{P}_{[a,b]}\} = \{P' \cup P'' : P' \in \mathcal{P}_{[a,c]},\, P'' \in \mathcal{P}_{[c,b]}\}.$$

In particular,

$$\forall P \in \mathcal{P}_{[a,b]} \quad \exists P' \in \mathcal{P}_{[a,c]},\, P'' \in \mathcal{P}_{[c,b]} \quad \text{such that} \quad P_c = P' \cup P''$$

and vice versa

$$\forall P' \in \mathcal{P}_{[a,c]},\, P'' \in \mathcal{P}_{[c,b]} \quad \text{we have} \quad P' \cup P'' \in \{P_c : P \in \mathcal{P}_{[a,b]}\}.$$

Finally, for $P_c = P' \cup P''$,

$$L(f,P_c) = L(f,P') + L(f,P''), \qquad U(f,P_c) = U(f,P') + U(f,P''),$$

and $\quad U(f,P_c) - L(f,P_c) = \big(U(f,P') - L(f,P')\big) + \big(U(f,P'') - L(f,P'')\big).$

Properties of the Riemann integral.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Theorem. Fix $c \in (a, b)$. A bounded function $f(x)$ is Riemann integrable on $[a, b]$ if and only if $f(x)$ is Riemann integrable on $[a, c]$ and $[c, b]$. In the later case,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

By **convention**, for a Riemann integrable $f(x)$ on $[a, b]$, we set

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Then, for any $a, b, c \in \mathbb{R}$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

whenever the above integrals exist.

Properties of the Riemann integral.

Proposition. If $f(x)$ is Riemann integrable on $[a,b]$ and $c \in \mathbb{R}$, then $c\,f(x)$ is Riemann integrable, and $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$.

Theorem. If $f(x)$and $g(x)$ are both Riemann integrable on $[a,b]$, then $f(x)+g(x)$ is Riemann integrable on $[a,b]$, and

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof: For any $\epsilon > 0$, there are partitions P' and P'' of $[a,b]$ satisfying

$0 \le U(f,P') - L(f,P') \le \epsilon/2$ and $0 \le U(g,P'') - L(g,P'') \le \epsilon/2$.

Hence, for $P = P' \cup P''$, a refinement to both P' and P'',

$$U(f+g,P) - L(f+g,P) \le U(f,P) - L(f,P) + U(g,P) - L(g,P) \le \epsilon/2 + \epsilon/2 = \epsilon.$$

Properties of the Riemann integral.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Proposition. If $f(x)$ is Riemann integrable on $[a, b]$ $(a < b)$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Proof: For any partition P of $[a, b]$, $m \leq m_k \leq M_k \leq M$ and

$$m \sum_{k=1}^{n} \triangle x_k \leq L(f,P) = \sum_{k=1}^{n} m_k \triangle x_k \leq \int_a^b f(x)\,dx \leq U(f,P) = \sum_{k=1}^{n} M_k \triangle x_k \leq M \sum_{k=1}^{n} \triangle x_k.$$

Corollary. If $f(x)$ is Riemann integrable on $[a, b]$ and $|f(x)| \leq M$, then

$$\left| \int_a^b f(x)\,dx \right| \leq M(b-a).$$

Properties of the Riemann integral.

Proposition. If $f(x)$ and $g(x)$ are Riemann integrable on $[a, b]$ and $f(x) \le g(x)$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof: For any partition P of $[a, b]$, $U(f, P) \le U(g, P)$.

Proposition. If $f(x)$ is Riemann integrable on $[a, b]$ and then so is $|f(x)|$, and

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Proof:

Let $m'_k = \text{inf}\{|f(x)| : x \in [x_{k-1}, x_k]\}$ and $M'_k = \text{sup}\{|f(x)| : x \in [x_{k-1}, x_k]\}$.

The integrability follows from

$M'_k - m'_k \le M_k - m_k$ yielding $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$.

Finally, $f(x) \le |f(x)|$ and $-f(x) \le |f(x)|$ yield the inequality.

The Fundamental Theorem of Calculus.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Fundamental Theorem of Calculus. (i). Suppose function $f(x)$ is Riemann integrable on $[a, b]$ and let $F(x)$ be its antiderivative, i.e., $F'(x) = f(x)$ for all $x \in [a, b]$. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \Leftrightarrow \quad \int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

(ii). Suppose function $f(x)$ is Riemann integrable on $[a, b]$ and let

$$F(x) = \int_a^x f(t)\,dt \qquad \forall x \in [a, b].$$

Then $F(x)$ is continuous on $[a, b]$ and $F'(y) = f(y)$ whenever $f(x)$ is continuous at $y \in [a, b]$.

The Fundamental Theorem of Calculus.

Proof (cont.): (i). For any partition P of $[a,b]$, the Mean Value Theorem yields

$$L(f,P) \leq F(b) - F(a) = \sum_{k=1}^{n} \Big(F(x_k) - F(x_{k-1}) \Big) = \sum_{k=1}^{n} f(x_k^*) \triangle x_k \leq U(f,P)$$

where $x_k^* \in (x_{k-1}, x_k)$. Thus, $L(f) \leq F(b) - F(a) \leq U(f)$, where

$$L(f) = U(f) = \int_a^b f(x)\,dx.$$

(ii). Since $|f(x)| \leq M$, $|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \leq M\,|y-x|$

yielding the continuity of $F(x)$ on $[a,b]$.

Now, if $f(x)$ is continuous at $y \in [a,b]$, then

$\forall \epsilon > 0\ \exists \delta > 0$ such that $|f(t) - f(y)| < \epsilon$ whenever $|t-y| < \delta$.

Hence, for $x \neq y$ in $[a,b]$ satisfying $|x-y| < \delta$,

$$\left| \frac{F(x) - F(y)}{x-y} - f(y) \right| = \frac{1}{|x-y|} \left| \int_x^y \Big(f(t) - f(y) \Big)\,dt \right| \leq \epsilon.$$

Mean Value Theorem for Definite Integrals.

Recall the following generalized Mean Value Theorem.

Cauchy Mean Value Theorem. If $f(x)$ and $g(x)$ are both continuous on $[a,b]$ $(a < b)$ and differentiable on (a,b), then there exists some $c \in (a,b)$ such that

$$\Big(f(b) - f(a)\Big)g'(c) = \Big(g(b) - g(a)\Big)f'(c).$$

Next, we prove the Mean Value Theorem for Definite Integrals.

Mean Value Theorem for Definite Integrals. If $f(x)$ is continuous on $[a,b]$ $(a < b)$ and if $g(x)$ is an integrable nonnegative function on $[a,b]$, then there exists $c \in (a,b)$ such that

$$\int\limits_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Mean Value Theorem for Definite Integrals.

Mean Value Theorem for Definite Integrals. If $f(x)$ is continuous on $[a,b]$ $(a<b)$ and if $g(x)$ is an integrable nonnegative function on $[a,b]$, then there exists $c \in (a,b)$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Proof for $g>0$: Suppose $g(x)>0$ for all $x \in [a,b]$. Consider functions

$$\Phi(x) = \int_a^x f(t)g(t)\,dt \quad \text{and} \quad G(x) = \int_a^x g(t)\,dt \qquad \forall x \in [a,b].$$

Then, by Cauchy Mean Value Theorem and the Fundamental Theorem of Calculus, there exists $c \in (a,b)$ such that

$$g(c)\int_a^b f(x)g(x)dx = \big(\Phi(b)-\Phi(a)\big)G'(c) = \big(G(b)-G(a)\big)\Phi'(c) = f(c)g(c)\int_a^b g(x)dx.$$

Mean Value Theorem for Definite Integrals.

Mean Value Theorem for Definite Integrals. If $f(x)$ is continuous on $[a, b]$ $(a < b)$ and if $g(x)$ is an integrable nonnegative function on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\int\limits_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Proof: W.l.o.g. assume $\int_a^b g(x)\,dx > 0$. Let $w(x) = \frac{g(x)}{\int_a^b g(x)\,dx}$.

Then, $\int\limits_a^b w(x)\,dx = 1$ and $\min\limits_{x\in[a,b]} f(x) \le \int\limits_a^b f(x)w(x)\,dx \le \max\limits_{x\in[a,b]} f(x)$.

Hence, by the Intermediate Value Theorem there exists $c \in (a, b)$ such that

$$\int\limits_a^b f(x)w(x)\,dx = f(c).$$

Integrable Limit Theorem.

Integrable Limit Theorem. Suppose sequence of Riemann integrable functions $f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$ $(a < b)$. Then, $f(x)$ is Riemann integrable and

$$\lim_{n\to\infty} \int\limits_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof: For a given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N \quad \forall x \in [a, b] \quad |f(x) - f_n(x)| < \epsilon/(b-a)$$

and

$$\left| \int\limits_a^b f(x)\,dx - \int\limits_a^b f_n(x)\,dx \right| \leq \int\limits_a^b |f(x) - f_n(x)|\,dx \leq \frac{\epsilon}{b-a}(b-a) = \epsilon.$$

Integrating functions with discontinuities.

For $c \in [a, b]$, let $f_c(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$. Then, for a given $\delta > 0$ and a partition P_δ of $[a, b]$ with $\triangle x_k = x_k - x_{k-1} \leq \delta$ for all k,

$$0 \leq L(f_c) \leq U(f_c) \leq U(f_c, P_\delta) \leq 2\delta.$$

Hence, $f_c(x)$ is Riemann integrable on $[a, b]$, and

$$\int_a^b f_c(x)\, dx = 0.$$

Next, consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Theorem. If $f(x)$ is Riemann integrable on $[a', b]$ for all $a' \in (a, b)$, then $f(x)$ is Riemann integrable on $[a, b]$.

Analogously, if $f(x)$ is Riemann integrable on $[a, b']$ for all $b' \in (a, b)$, then $f(x)$ is Riemann integrable on $[a, b]$.

The above theorem implies that a bounded function on $[a, b]$ with a single discontinuity is Riemann integrable

Integrating functions with discontinuities.

Consider a bounded function $f(x)$ on a closed interval $[a, b]$.

Theorem. If $f(x)$ is Riemann integrable on $[a', b]$ for all $a' \in (a, b)$, then $f(x)$ is Riemann integrable on $[a, b]$.

Proof: There is $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$. For $\epsilon > 0$, pick $a' \in (a, b)$ such that $a' - a < \frac{\epsilon}{4M}$. Since $f(x)$ is Riemann integrable on $[a', b]$, there is a partition P' of $[a', b]$ such that

$$0 \le U(f, P') - L(f, P') < \frac{\epsilon}{2}.$$

Next, consider partition $P = \{a\} \cup P'$ of $[a, b]$. Then,

$$0 \le U(f, P) - L(f, P) \le 2M(a' - a) + U(f, P') - L(f, P') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $f(x)$ is Riemann integrable on $[a, b]$.

Improper integral.

Definition. Suppose $f(x)$ is defined on $[a, b)$ and integrable on every interval $[a, c]$ with $c \in (a, b)$. Then, the improper integral on $[a, b)$ is defined as $\int_a^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(x)\,dx$.

Similarly, for $f(x)$ is defined on $(a, b]$, $\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx$.

If F is an antiderivative of f, i.e., $F'(x) = f(x)$, then the Fundamental Theorem of Calculus implies

$$\int_a^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(x)\,dx = \lim_{c \to b-} F(c) - F(a).$$

Example.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 - \lim_{c \to 0+} 2\sqrt{c} = 2.$$

Improper integral.

Definition. Suppose $f(x)$ is defined on $[a,\infty)$ and integrable on every interval $[a,b]$ $(b>a)$. Then, the improper integral on $[a,\infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Similarly, $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.

If F is an antiderivative of f, i.e., $F'(x)=f(x)$, then the Fundamental Theorem of Calculus implies

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty} F(b)-F(a).$$

Example.

$$\int_0^\infty e^{-x}\,dx = -\lim_{b\to\infty} e^{-b} + e^0 = e^0 = 1.$$