MTH 312

Lectures 8-9

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Topics:

- Taylor polynomial approximation.
- Taylor series.
- Error in approximation.
- Lagrange's Remainder Theorem.
- Integral form of the remainder.

Taylor polynomial approximation.

We approximate functions by Taylor polynomials around x_0.

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2$$

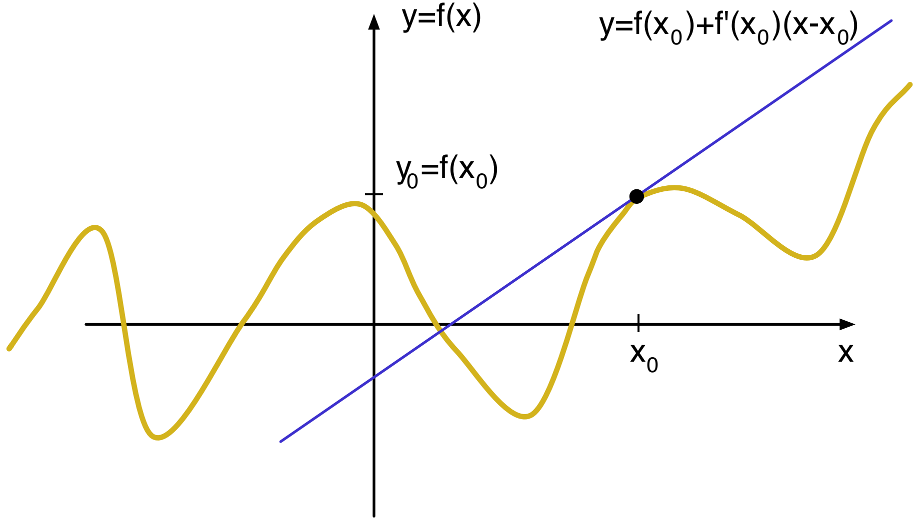
$$T_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3$$

$\dots$

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$, ($f^{(k)}$ is kth derivative),

is the n**th Taylor polynomial of** $f(x)$ **around** x_0**.**

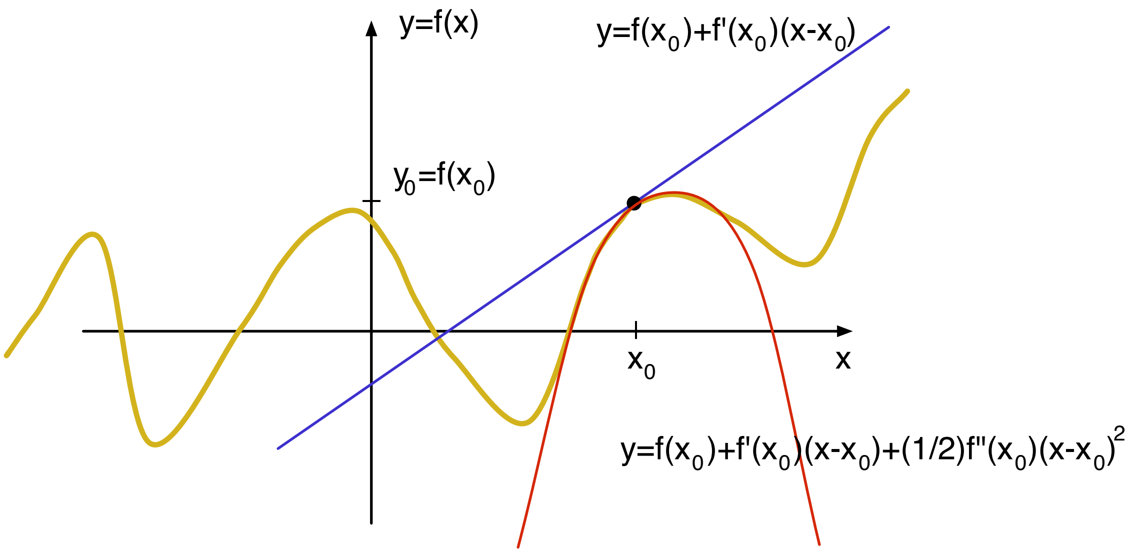
Linear approximation.



Linear approximation of function $y = f(x)$ around x_0 is given by the **first Taylor polynomial**

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

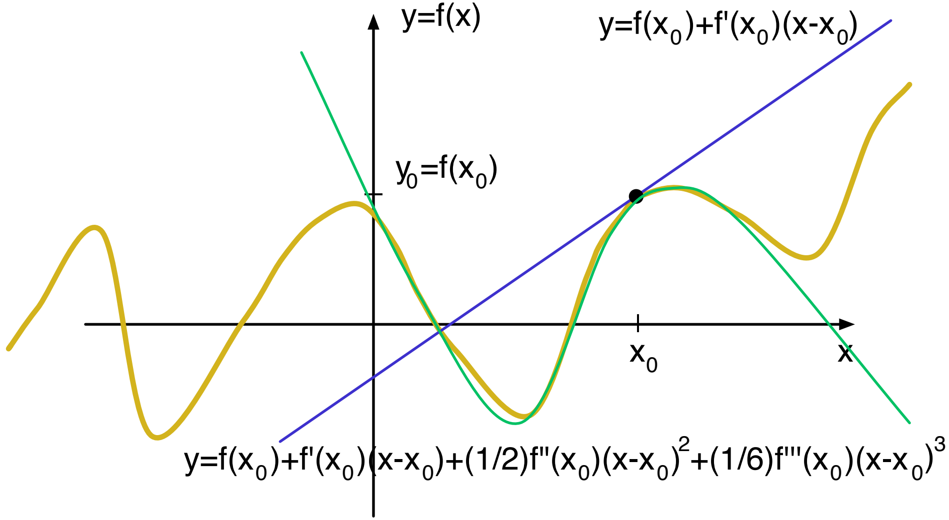
Quadratic approximation.



Quadratic approximation of function $y = f(x)$ around x_0 is given by **the second Taylor polynomial**

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2$$

3rd Taylor polynomial approximation.



Third order approximation around x_0 is given by **the 3rd Taylor polynomial**

$$T_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3$$

Derivatives.

Assume $f(x)$ has derivatives up to order n at x_0. Then $T_n(x)$ is **the only polynomial of degree** $\leq n$ such that

$$T_n(x_0) = f(x_0)$$

$$T_n'(x_0) = f'(x_0)$$

$$T_n''(x_0) = f''(x_0)$$

$$\dots$$

$$T_n^{(n)}(x_0) = f^{(n)}(x_0)$$

Check it is true for $T_3(x)$:

$$T_3(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\frac{1}{3!}f'''(x_0)(x-x_0)^3$$

$$T_3'(x)=0+f'(x_0)\cdot 1+\frac{2}{2!}f''(x_0)(x-x_0)+\frac{3}{3!}f'''(x_0)(x-x_0)^2$$

and therefore

$$T_3'(x)=f'(x_0)+f''(x_0)(x-x_0)+\frac{1}{2!}f'''(x_0)(x-x_0)^2$$

Now, $T_3''(x)=0+f''(x_0)\cdot 1+\frac{2}{2!}f'''(x_0)(x-x_0)=f''(x_0)+f'''(x_0)(x-x_0)$

and $T_3'''(x)=f'''(x_0)$.

Thus $T_3(x_0)=f(x_0)$, $T_3'(x_0)=f'(x_0)$, $T_3''(x_0)=f''(x_0)$

and $T_3'''(x_0)=f'''(x_0)$.

Example. Let $f(x) = e^x$ and $x_0 = 0$.

Now $f'(x) = e^x$, $f''(x) = e^x, \ldots, f^{(n)}(x) = e^x$,

$$T_1(x) = e^0 + e^0 x = 1 + x = \sum_{k=0}^{1} \frac{x^k}{k!}$$

$$T_2(x) = e^0 + e^0 x + \frac{1}{2!} e^0 x^2 = 1 + x + \frac{1}{2!} x^2 = \sum_{k=0}^{2} \frac{x^k}{k!}$$

$$T_3(x) = e^0 + e^0 x + \frac{1}{2!} e^0 x^2 + \frac{1}{3!} e^0 x^3 = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 = \sum_{k=0}^{3} \frac{x^k}{k!}$$

$\ldots$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots + \frac{1}{n!} x^n = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Taylor series.

The nth Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

of $f(x)$ around $x_0 = 0$ is the nth partial sum of the infinite power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

called the **Taylor series** of $f(x)$ around $x_0 = 0$.

Example. Consider $f(x)=\frac{1}{1-x}$. We find its Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ around $x_0 = 0$. Here

$f(x)=(1-x)^{-1}$ and $f(0)=1=0!$

$f'(x)=1\cdot(1-x)^{-2}$ and $f'(0)=1=1!$

$f''(x)=1\cdot 2\cdot(1-x)^{-3}$ and $f''(0)=1\cdot 2=2!$

$f'''(x)=1\cdot 2\cdot 3\cdot(1-x)^{-4}$ and $f'''(0)=1\cdot 2\cdot 3=3!$

$\dots$

$f^{(k)}(x)=1\cdot 2\cdot 3\cdot\dots\cdot k\cdot(1-x)^{-k-1}$ and $f^{(k)}(0)=1\cdot 2\cdot 3\cdot\dots\cdot k=k!$

Therefore $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty} \frac{k!}{k!}x^k=\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}=f(x)$

whenever $|x| < 1$.

When Taylor series converges to $f(x)$.

Taylor polynomial $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ is the nth partial sum of $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, and

$$f(x) = T_n(x) + R_n(x)$$

Thus, the limit $\lim_{n\to\infty} T_n(x) = f(x)$ if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

Lagrange: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ for some c between $x_0 = 0$ and x.

So $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x)$ if and only if $\lim_{n\to\infty} R_n(x) = 0$.

Lagrange: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ for some c between 0 and x.

Example. Show that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x.

First, $R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$ for some c between 0 and x.

Here $0 < e^c < e^{|x|}$ and

$$|R_n(x)| = \frac{e^c |x|^{n+1}}{(n+1)!} \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, since $\lim_{m\to\infty} \frac{|x|^m}{m!} = 0$ for any given x.

Thus $\lim_{n\to\infty} R_n(x) = 0$ and $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x.

Lagrange's Remainder Theorem.

Lagrange's Remainder Theorem. Suppose $f(x)$ is $n+1$ times differentiable in $(-R, R)$, where R is the radius of convergence of its Taylor series. Then,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Proof: W.l.o.g. suppose $x > 0$, by the Cauchy Mean Value Theorem also known as Generalized Mean Value Theorem,

$$\frac{R_n(x)}{x^{n+1}/(n+1)!} = \left.\frac{R_n'(t)}{(t^{n+1}/(n+1)!)'}\right|_{t=x_1} = \frac{R_n'(x_1)}{x_1^n/n!} \quad \text{for } 0 < x_1 < x$$

$$\frac{R_n(x)}{x^{n+1}/(n+1)!} = \frac{R_n'(x_1)}{x_1^n/n} = \frac{R_n''(x_2)}{x_2^{n-1}/(n-1)!} \quad \text{for } 0 < x_2 < x_1 < x$$

$$\vdots$$

$$\frac{R_n(x)}{x^{n+1}/(n+1)!} = \frac{R_n^{(k)}(x_k)}{x_k^{n-k+1}/(n-k+1)!} \quad \text{for } 0 < x_k < \ldots < x_2 < x_1 < x$$

So, $$\frac{R_n(x)}{x^{n+1}/(n+1)!} = R_n^{(n+1)}(x_n) = f^{(n+1)}(x_n) \quad \text{for } 0 < x_n < \ldots < x_2 < x_1 < x.$$

Lagrange's Remainder Theorem.

Cauchy Mean Value Theorem. If $f(x)$ and $g(x)$ are both continuous on $[a, b]$ $(a < b)$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

Thus, if $g(a) \neq g(b)$ and $g'(c) \neq 0$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof: Let

$$h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x),$$

then, by the Mean Value Theorem, $\exists c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = 0.$$

Hence,

$$\big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0.$$

Taylor series around $x_0 = 0$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \quad = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \quad = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{for all } x$$

The Taylor series of $f(x)$ around $x_0 = 0$ is called the **Maclaurine series** of $f(x)$.

Integral form of the remainder.

Integral Remainder Theorem.

$$R_n(x) = \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

Proof: We use integration by parts recursively.

$$\int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt = f^{(n)}(t)\frac{(x-t)^n}{n!}\Big|_{t=0}^{x} + \int_0^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

$$= -f^{(n)}(0)\frac{x^n}{n!} + \int_0^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

$$= -f^{(n)}(0)\frac{x^n}{n!} - f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + \int_0^x f^{(n-1)}(t)\frac{(x-t)^{n-2}}{(n-2)!}\,dt$$

$$\ldots = -\sum_{k=1}^{n} f^{(k)}(0)\frac{x^k}{k!} + \int_0^x f'(t)\,dt = -T_n(x) + f(x) = R_n(x).$$

Integral form of the remainder.

Integral Remainder Theorem.

$$R_n(x) = \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

Mean Value Theorem for Definite Integrals. If $f(x)$ is continuous on $[a,b]$ $(a<b)$ and if $g(x)$ is an integrable nonnegative function on $[a,b]$, then there exists $c \in (a,b)$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Together, the Integral Remainder Theorem and the Mean Value Theorem for Definite Integrals yield

Lagrange's Remainder Theorem. Suppose $f(x)$ is $n+1$ times differentiable in $(-R,R)$, where R is the radius of convergence of its Taylor series. Then,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Other base points.

In general, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

are called the **Taylor series** of $f(x)$ around x_0.

Example. Consider $f(x) = e^x$ and $x_0 = 1$.

Here $f^{(k)}(x) = e^x$ and $f^{(k)}(x_0) = e^{x_0} = e$.

Therefore, $e^x = \sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k$, where convergence of the Taylor series to e^x can be shown for all real values x using Lagrange's formula.

Error in approximation.

Error (remainder) is defined as

$$R_n(x) = f(x) - T_n(x)$$

Integral Remainder Theorem.

$$R_n(x) = \int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

Lagrange's Remainder Theorem:

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-x_0)^{n+1}$$

for some c between x and x_0.

$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1}$ for some c between x and x_0.

Example. For $-1 \le x \le 1$, $f(x) = e^x$ and $x_0 = 0$,

$$R_4(x) = \frac{1}{5!} e^c x^5$$

for some c between x and 0.

Thus $|R_4(x)| = \frac{1}{5!} e^c |x|^5 \le \frac{e^1}{5!} = \frac{e}{120}$.

Hence $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + R_4(x)$,

where $|R_4(x)| \le \frac{e}{120} \approx 0.02265$ when $-1 \le x \le 1$.