MTH 312

Lectures 4-7

Yevgeniy Kovchegov
Oregon State University

- Series of functions.
- Weierstrass M-Test.
- $\liminf$ and $\limsup$
- Power series.

Series of functions.

Consider a sequence of functions $\{f_n\}$, $n \in \mathbb{N}$, defined on a set $D \subseteq \mathbb{R}$.

Definition. The series of functions $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function s on D if the sequence of its partial sums $s_n = \sum_{k=1}^{n} f_k$ converges pointwise to s on D.

If $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function s, we write $s = \sum_{k=1}^{\infty} f_k$.

Definition. The series of functions $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function s on D if the sequence of its partial sums $s_n = \sum_{k=1}^{n} f_k$ converges uniformly to s on D.

Series of functions.

Consider a sequence of functions $\{f_n\}$, $n \in \mathbb{N}$, defined on a set $D \subseteq \mathbb{R}$. Let $s_n = \sum_{k=1}^{n} f_k$ denote the corresponding sequence of partial sums for the series of functions $\sum_{k=1}^{\infty} f_k$. Recall

Continuous Limit Theorem. Suppose sequence $\{f_n\}$ converges uniformly to f on D. Then, the following holds for any given $x_0 \in D$:

if f_n are continuous at x_0, then f is continuous at x_0.

Applying the above Continuous Limit Theorem to the sequence of functions s_n yields the following result.

Theorem. Suppose series of functions $\sum_{k=1}^{\infty} f_k$ converges uniformly to s on D. Then, the following holds for any given $x_0 \in D$:

if f_n are continuous at x_0, then $s = \sum_{k=1}^{\infty} f_k$ is continuous at x_0.

Series of functions.

Consider a sequence of functions $\{f_n\}$, $n \in \mathbb{N}$, defined on a closed interval $D = [a, b] \subset \mathbb{R}$. Let $s_n = \sum_{k=1}^{n} f_k$ denote the corresponding sequence of partial sums for the series of functions $\sum_{k=1}^{\infty} f_k$. Recall

Differentiable Limit Theorem. Suppose sequence of functions $\{f_n\}$ converges pointwise to f and sequence $\{f_n'\}$ converges uniformly to g. Then, function f is differentiable on $[a, b]$ and $f' = g$.

Applying the above Differentiable Limit Theorem to the sequence of functions s_n yields the following result.

Theorem. Suppose series of functions $\sum_{k=1}^{\infty} f_k$ converges pointwise to s and series $\sum_{k=1}^{\infty} f_k'$ converges uniformly to h on $D = [a, b]$. Then, $s = \sum_{k=1}^{\infty} f_k$ is differentiable on $[a, b]$ and $s' = h$, i.e.,

$$\left(\sum_{k=1}^{\infty} f_k(x) \right)' = \sum_{k=1}^{\infty} f_k'(x) \qquad \forall x \in [a, b].$$

Weierstrass M-test.

Consider a sequence of functions $\{f_n\}$, $n \in \mathbb{N}$, defined on a set $D \subseteq \mathbb{R}$. Let $s_n = \sum_{k=1}^{n} f_k$ denote the corresponding sequence of partial sums for the series of functions $\sum_{k=1}^{\infty} f_k$.

Weierstrass M-test. Suppose there exists a sequence $\{M_n\}$ satisfying $\forall n \in \mathbb{N}\ \forall x \in D \quad |f_n(x)| \leq M_n$. Then, if the series $\sum_{k=1}^{\infty} M_k$ converges, then the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D.

Proof: The sequence of partial sums $\mu_n = \sum_{k=1}^{n} M_k$ is Cauchy:

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N \quad |\mu_n - \mu_m| < \epsilon,$$

Thus, by Cauchy Criterion for Uniform Convergence, the sequence $\{s_n\}$ converges uniformly on D as

$$\forall n, m \geq N\ \forall x \in D \quad |s_n(x) - s_m(x)| \leq |\mu_n - \mu_m| < \epsilon,$$

and therefore, the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D.

Geometric series.

Geometric series: $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$

Lemma. For $x \neq 1$,

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Proof: $(1-x)(1+x+x^2+x^3+\cdots+x^n)=[1+x+x^2+x^3+\cdots+x^n]-[x+x^2+x^3+\cdots+x^n+x^{n+1}]=1-x^{n+1}$

Summing the geometric series:

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & \text{pointwise on } (-1,1), \\ \text{diverges} & \text{on } (-\infty,-1] \cup [1,\infty). \end{cases}$$

Geometric series.

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & \text{pointwise on } (-1,1), \\ \text{diverges} & \text{on } (-\infty,-1] \cup [1,\infty). \end{cases}$$

Lemma. $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$ pointwise on $(-1,1)$.

Proof: Fix $r \in (0,1)$, then for all $x \in [-r,r]$ and all $k \in \mathbb{N}$, $|kx^{k-1}| \leq M_k = kr^{k-1}$, where

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} kr^{k-1} \quad \text{converges by Ratio Test.}$$

Hence, by Weierstrass M-test, series $\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly on $[-r,r]$. Therefore,

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (x^k)' = \left(\sum_{k=0}^{\infty} x^k \right)' = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2} \qquad \forall x \in [-r,r].$$

lim inf **and** lim sup

Consider a sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$.

Definition. The limits defined as

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{k:\,k\geq n} a_k\right) \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{k:\,k\geq n} a_k\right)$$

are called the limit infimum and limit supremum of the sequence $\{a_n\}$.

Lemma. If the sequence $\{a_n\}$ is bounded from above and from below, then the limits $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ exist and are finite.

Lemma. The limit $\lim_{n\to\infty} a_n$ exists (or equals $\pm\infty$) if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

In which cased

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

lim inf **and** lim sup

Consider a sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$. Denote

$$\mu_n = \inf_{k\colon k \geq n} a_k \quad \text{and} \quad \sigma_n = \sup_{k\colon k \geq n} a_k.$$

Then,

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \mu_n \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n$$

Lemma. If the sequence $\{a_n\}$ is bounded from above and from below, then the limits $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ exist and are finite.

Proof: Observe that $\{\mu_n\}$ is a nondecreasing sequence bounded from above and $\{\sigma_n\}$ is a nonincreasing sequence bounded from below.

Lemma. The limit $\lim_{n\to\infty} a_n$ exists (or equals $\pm\infty$) if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

In which cased

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

$\liminf$ and $\limsup$

Consider a sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$. Denote

$$\mu_n = \inf_{k:\, k\geq n} a_k \quad \text{and} \quad \sigma_n = \sup_{k:\, k\geq n} a_k.$$

Then,

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \mu_n \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n$$

Lemma. The limit $\lim_{n\to\infty} a_n$ exists (or equals $\pm\infty$) if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

In which cased

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof: Suppose $\lim_{n\to\infty} a_n = L < \infty$, then

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n \geq N \quad L - \epsilon \leq a_n \leq L + \epsilon$$

$$\text{and therefore} \quad \forall n \geq N \quad L - \epsilon \leq \mu_n \leq a_n \leq \sigma_n \leq L + \epsilon$$

$$\text{yielding} \quad L - \epsilon \leq \lim_{n\to\infty} \mu_n \leq L \leq \lim_{n\to\infty} \sigma_n \leq L + \epsilon$$

Hence, $\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n$.

The converse follows from $\mu_n \leq a_n \leq \sigma_n$ and the Squeeze Theorem.

lim inf **and** lim sup

Example. Let $a_n = \begin{cases} -n & \text{if } n \text{ is even} \\ 1 + \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$. Then,

$$\liminf_{n\to\infty} a_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} a_n = 1.$$

Example. Let q_n be an enumeration of all rational numbers in $(0,1)$ (recall they countable!). Then,

$$\liminf_{n\to\infty} q_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} q_n = 1.$$

lim inf **and** lim sup

Consider a sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$. Denote

$$\mu_n = \inf_{k\colon k \geq n} a_k \quad \text{and} \quad \sigma_n = \sup_{k\colon k \geq n} a_k.$$

Theorem. There exists a subsequence $\{a_{n_k}\}$ satisfying

$$\lim_{k \to \infty} a_{n_k} = \limsup_{n \to \infty} a_n.$$

The analogous statement holds for $\liminf\limits_{n \to \infty} a_n$.

Proof: First, there exists $n_1 \in \mathbb{N}$ satisfying

$$\sigma_1 - \frac{1}{2} \leq a_1 \leq \sigma_1$$

Iteratively, for each $k \geq 2$, one can find $n_k > n_{k-1}$ satisfying

$$\sigma_{n_{k-1}+1} - \frac{1}{2^k} \leq a_{n_k} \leq \sigma_{n_{k-1}+1}.$$

Next, apply the Squeeze Theorem.

lim inf **and** lim sup

Consider a sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$. Denote

$$\mu_n = \inf_{k:\, k \geq n} a_k \quad \text{and} \quad \sigma_n = \sup_{k:\, k \geq n} a_k.$$

Lemma. Denote $\overline{L} = \limsup_{n\to\infty} a_n$. For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N \quad a_n < \overline{L} + \epsilon.$$

The analogous statement holds for $\underline{L} = \liminf_{n\to\infty} a_n$.

Proof: Since $\overline{L} = \lim_{n\to\infty} \sigma_n$, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N \quad a_n \leq \sigma_n < \overline{L} + \epsilon.$$

Power series.

Definition. A series of functions of the kind

$$\sum_{k=0}^{\infty} a_k\, x^k, \text{ where } a_k \in \mathbb{R},$$

are called power series.

Definition. A real number $R \geq 0$ is said to be the radius of convergence of a power series $\sum_{k=0}^{\infty} a_k\, x^k$ if the series converges absolutely for $x \in (-R, R)$ and diverges for $x \in (-\infty, -R) \cup (R, \infty)$.

Theorem. The radius of convergence R of a power series $\sum_{k=0}^{\infty} a_k\, x^k$ can be expressed as

$$R = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}, \qquad \text{where } \frac{1}{0} = \infty.$$

Power series.

Theorem.

$$R = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}, \qquad \text{where } \frac{1}{0} = \infty.$$

Proof: Let $R = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}$.

(i) Suppose $x \in (-R, R)$. Then

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} < \frac{1}{|x|}.$$

and for $\epsilon \in \left(0, \frac{1}{|x|} - \frac{1}{R}\right)$,

$$\exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n \geq N \quad |a_n|^{1/n} < \frac{1}{R} + \epsilon \ < \frac{1}{|x|}$$

$$\text{implying} \quad |a_n x^n| < \alpha^n, \quad \text{where} \quad \alpha = |x| \left(\frac{1}{R} + \epsilon\right) \in (0, 1).$$

Hence, $\sum\limits_{n=0}^{\infty} a_n x^n$ converges absolutely for $x \in (-R, R)$

Power series.

Theorem.

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}, \qquad \text{where } \frac{1}{0} = \infty.$$

Proof(cont.): (ii) Suppose

$$|x| > R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

Then there exists a subsequence $\{a_{n_k}\}$ s.t.

$$\exists \text{ subsequence } \{a_{n_k}\} \quad \text{s.t.} \quad \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{k\to\infty} \left|a_{n_k}\right|^{1/n_k}.$$

Thus, $1 < \frac{|x|}{R} = \lim_{k\to\infty} \left|a_{n_k}\, x^{n_k}\right|^{1/n_k}$ and

$$\exists K \in \mathbb{N} \quad \text{s.t.} \quad \forall k \geq K \quad \left|a_{n_k}\, x^{n_k}\right| > 1$$

Hence, $\lim_{n\to\infty} a_n\, x^n \neq 0$, and $\sum_{n=0}^{\infty} a_n\, x^n$ diverges by the Basic Divergence Test.

Power series.

Definition. A real number $R \geq 0$ is said to be the radius of convergence of a power series $\sum_{k=0}^{\infty} a_k x^k$ if the series converges absolutely for $x \in (-R, R)$ and diverges for $x \in (-\infty, -R) \cup (R, \infty)$.

Theorem.

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}, \qquad \text{where } \frac{1}{0} = \infty.$$

Example. For the power series $\sum_{k=0}^{\infty} x^k$, the radius of convergence $R = 1$ and the interval of convergence is $(-1, 1)$.

Example. For the power series $\sum_{k=0}^{\infty} \frac{1}{k} x^k$, the radius of convergence $R = 1$ and the interval of convergence is $[-1, 1)$.

Example. For the power series $\sum_{k=0}^{\infty} \frac{1}{k^2} x^k$, the radius of convergence $R = 1$ and the interval of convergence is $[-1, 1]$.

Power series.

Theorem. Let R denote the radius of convergence of a power series $\sum_{k=0}^{\infty} a_k\, x^k$. Then for any $r \in (0, R)$, the power series converges uniformly on $[-r, r]$.

Proof: Select $\alpha \in \left(\frac{r}{R}, 1\right)$. Then, $\forall x \in [-r, r]$,

$$\limsup_{n\to\infty} |a_n\, x^n|^{1/n} = \frac{|x|}{R} \le \frac{r}{R} < \alpha < 1.$$

Take $\epsilon = \alpha - \frac{r}{R} > 0$. Then

$$\forall x \in [-r, r] \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n \ge N \quad |a_n\, x^n|^{1/n} < \frac{|x|}{R} + \epsilon \le \alpha$$

Thus,

$$|a_n\, x^n| < \alpha^n, \qquad \text{where} \quad \sum_{n=0}^{\infty} \alpha^n < \infty$$

as $0 < \alpha < 1$. Hence, by the Weierstrass M-Test, $\sum_{k=0}^{\infty} a_k\, x^k$ converges uniformly on $[-r, r]$.

Power series.

Lemma. Let R denote the radius of convergence of a power series $\sum_{k=0}^{\infty} a_k\, x^k$. Then

$$\sum_{m=0}^{\infty}(m+1)a_{m+1}\, x^m = \sum_{k=0}^{\infty}\left(a_k\, x^k\right)' = \left(\sum_{k=0}^{\infty} a_k\, x^k\right)'$$

Proof: Notice that the radius of convergence R' of $\sum_{m=0}^{\infty}(m+1)a_{m+1}\, x^m$ equals R. That is

$$\frac{1}{R'} = \limsup_{n\to\infty}\left|(n+1)a_{n+1}\right|^{1/n} = \limsup_{n\to\infty}\left|a_{n+1}\right|^{1/n} = \limsup_{n\to\infty}\left|a_n\right|^{1/n} = \frac{1}{R}.$$

Thus, for any $r \in (0, R)$, the power series $\sum_{k=0}^{\infty}\left(a_k\, x^k\right)'$ converges uniformly on $[-r, r]$.

Power series.

Limit Root Test. Consider series series $\sum_{k=0}^{\infty} a_k$. Let

$$L = \limsup_{n\to\infty} \left|a_n\right|^{1/n}.$$

Then, (a) if $L < 1$ the series $\sum_{k=0}^{\infty} a_k$ converges absolutely, and (b) if $L > 1$ the series $\sum_{k=0}^{\infty} a_k$ diverges.

Proof: Let R denote the radius of convergence of $\sum_{k=0}^{\infty} a_k\, x^k$. Then, $R = 1/L$. If $L < 1$, then $R > 1$ and the power series $\sum_{k=0}^{\infty} a_k\, x^k$ converges absolutely at $x = 1$.

If $L > 1$, then $R < 1$ and the power series $\sum_{k=0}^{\infty} a_k\, x^k$ diverges at $x = 1$.

Exponential series.

Let

$$e(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots$$

Notice that $e(0) = 1$ and $e(1) = e$. We show that

- $e(x)e(y) = e(x + y)$
- $e'(x) = e(x)$

Here, $e(x) = e^x$.

Review of series.

Geometric series.

- If $|x| < 1$, the series $\sum_{k=0}^{\infty} x^k$ converges and

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

- If $|x| \geq 1$, the series $\sum_{k=0}^{\infty} x^k$ diverges.

Basic divergence test.

$$\text{If } \lim_{k\to\infty} a_k \neq 0 \text{ then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

The integral test.

Let $f(x)$ be a positive, decreasing, and continuous for $x \geq 1$. Let $a_k = f(k)$. Then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ converges}$$

The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$

Basic comparison test.

If $0 \leq a_k \leq b_k$ for all k, then

$$\sum_{k=1}^{\infty} b_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \Rightarrow \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Limit comparison test.

If $a_k > 0$ and $b_k > 0$ for all large k and the limit

$$0 < \lim_{k\to\infty} \frac{a_k}{b_k} < \infty \quad \text{is positive an finite, then}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \sum_{k=1}^{\infty} b_k \text{ converges}$$

Alternating series test.

If $b_0 > b_1 > b_2 > \ldots$ and $b_k \to 0$, then the alternating series $\sum_{k=0}^{\infty} (-1)^k b_k$ converges.

Absolute convergence.

Fact: If an infinite series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then it must be **convergent** (in the ordinary sense).

The root test.

Let $a_k \geq 0$. Then

$$(a_k)^{1/k} \leq r \quad \text{for all large } k \quad \text{and some } r<1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{converges}$$

$$(a_k)^{1/k} \geq 1 \quad \text{for all large } k \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{diverges}$$

If $\lim_{k\to\infty} |a_k|^{1/k} = L$, finite or infinite, then

- $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $L < 1$
- $\sum_{k=1}^{\infty} a_k$ **diverges** if $L > 1$
- **unable to determine** using this test if $L = 1$

The ratio test.

Let $a_k > 0$. Then

$$\frac{a_{k+1}}{a_k} \le r \quad \text{for all large } k \quad \text{and some } r<1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{converges}$$

$$\frac{a_{k+1}}{a_k} \ge 1 \quad \text{for all large } k \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{diverges}$$

If $\lim_{k\to\infty} |a_k|^{1/k} = L$, finite or infinite, then

- $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $L < 1$
- $\sum_{k=1}^{\infty} a_k$ **diverges** if $L > 1$
- **unable to determine** using this test if $L = 1$

Example. Determine whether the following series converges or diverges:

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

The test(s) used:

basic divergence

integral test basic comparison

limit comparison

alternating series root test

ratio test

Example. Determine whether the following series converges or diverges:

$$\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k}$$

The test(s) used:

basic divergence

integral test   basic comparison

limit comparison

alternating series   root test

ratio test

Example. Determine whether the following series converges or diverges:

$$\sum_{k=1}^{\infty} (-1)^k \sqrt[k]{5}$$

The test(s) used:

basic divergence
integral test
basic comparison
limit comparison
alternating series
root test
ratio test

Example. Determine whether the following series converges or diverges:

$$\sum_{k=0}^{\infty} \frac{(-200)^k}{k!}$$

The test(s) used:

basic divergence

integral test

basic comparison

limit comparison

alternating series

root test

ratio test

Example. Find all x such that the following series converges:

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

The test(s) used:

basic divergence

integral test

basic comparison

limit comparison

alternating series

root test

ratio test

Example. Find all x such that the following series converges:

$$\sum_{k=1}^{\infty} kx^k$$

The test(s) used:

basic divergence
integral test
basic comparison
limit comparison
alternating series
root test
ratio test

Analytic properties of power series.

Example. Let $|x| < 1$. Then $\sum_{k=1}^{\infty} kx^k$ converges. Find its sum.

Solution:

$$\sum_{k=1}^{\infty} kx^k = \sum_{k=0}^{\infty} kx^k = x \cdot \sum_{k=0}^{\infty} kx^{k-1} = x \cdot \sum_{k=0}^{\infty} (x^k)'$$

$$= x \cdot \left(\sum_{k=0}^{\infty} x^k \right)' = x \cdot \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2}$$

So, $\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$.

Here is an **alternative approach:**

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots =$$

$$
\begin{array}{lllllll}
x + & x^2 + & x^3 + & x^4 + & x^5 + \cdots & = & \frac{x}{1-x} \\
 & + & + & + & + & & \\
 & x^2 + & x^3 + & x^4 + & x^5 + \cdots & = & \frac{x^2}{1-x} \\
 & & + & + & + & & \\
 & & x^3 + & x^4 + & x^5 + \cdots & = & \frac{x^3}{1-x} \\
 & & & + & + & & \\
 & & & x^4 + & x^5 + \cdots & = & \frac{x^4}{1-x} \\
 & & & & + & & \\
 & & & & x^5 + \ldots & & \vdots
\end{array}
$$

$$\frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} + \frac{x^4}{1-x} + \cdots = \frac{x}{1-x}(1 + x + x^2 + \ldots) = \frac{x}{(1-x)^2}$$

Analytic properties of power series.

Example. Let $|x| < 1$. Then $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges. Find its sum.

Solution:

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \int_0^x t^{k-1} dt = \int_0^x \left(\sum_{k=1}^{\infty} t^{k-1} \right) dt$$

$$= \int_0^x \frac{1}{1-t} dt = -\ln(1-x)$$

Example. Find all x such that the following series converges:

$$\sum_{k=2}^{\infty} k(k-1)x^k$$

The test(s) used:

basic divergence

integral test basic comparison

limit comparison

alternating series root test

ratio test

Example. Determine for what values of $p > 0$ the **alternating p-series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p}$$

is absolutely convergent, conditionally convergent, or divergent.

The test(s) used:

basic divergence

integral test

basic comparison

limit comparison

alternating series

root test

ratio test