

MTH 312  
Lectures 1-3

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- Sequences of functions.
- Pointwise convergence.
- Uniform convergence.
- Cauchy criterion for uniform convergence.
- Continuous limit theorem.
- Differentiable limit theorem.
- Series of functions.

## Sequences of functions. Pointwise convergence.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Definition.** The sequence of functions  $\{f_n\}$  converges **pointwise** to a function  $f$  if

$$\forall x \in D \quad \lim_{n \rightarrow \infty} f_n(x) = f(x).$$

Notations:  $\lim_{n \rightarrow \infty} f_n = f$  or  $f_n \rightarrow f$ .

**Question:** Suppose sequence  $\{f_n\}$  converges pointwise to a function  $f$ . If functions  $f_n(x)$  are continuous, would  $f$  necessarily be continuous?

**Example.** Let  $f_n(x) = x^n$  and  $D = [0, 1]$ . Then,  $\{f_n\}$  converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

## Pointwise convergence.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Definition.** The sequence of functions  $\{f_n\}$  converges **pointwise** to a function  $f$  if

$$\forall x \in D \quad \lim_{n \rightarrow \infty} f_n(x) = f(x).$$

Notations:  $\lim_{n \rightarrow \infty} f_n = f$  or  $f_n \rightarrow f$ .

**Question:** Suppose sequence  $\{f_n\}$  converges pointwise to a function  $f$ . If functions  $f_n(x)$  are differentiable and  $f(x)$  is continuous on  $D$ , would  $f(x)$  necessarily be differentiable?

**Example.** Let  $f_n(x) = x^{1+\frac{1}{2n-1}}$  and  $D = [-1, 1]$ . Then,  $\{f_n\}$  converges pointwise to  $f(x) = |x|$ .

## Uniform convergence.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Definition.** The sequence of functions  $\{f_n\}$  converges **uniformly** on  $D$  to a function  $f$  if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \forall x \in D \quad |f_n(x) - f(x)| < \epsilon.$$

**Easy to check:** Uniform convergence implies pointwise convergence.

**Cauchy Criterion for Uniform Convergence.** The sequence of functions  $\{f_n\}$  converges uniformly on  $D$  if and only if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N \forall x \in D \quad |f_n(x) - f_m(x)| < \epsilon,$$

i.e., sequence  $\{f_n\}$  is Cauchy.

**Proof:** Suppose sequence  $\{f_n\}$  is Cauchy, then for each  $x \in D$ , limit  $\lim_{n \rightarrow \infty} f_n(x)$  exists. Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Next we prove that  $\{f_n\}$  converges to  $f$  **uniformly**.

## Uniform convergence.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Cauchy Criterion for Uniform Convergence.** The sequence of functions  $\{f_n\}$  converges uniformly on  $D$  if and only if

$\forall \epsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n, m \geq N \forall x \in D \quad |f_n(x) - f_m(x)| < \epsilon$ ,  
i.e., sequence  $\{f_n\}$  is Cauchy.

**Proof:** Suppose sequence  $\{f_n\}$  is Cauchy, then for each  $x \in D$ , limit  $\lim_{n \rightarrow \infty} f_n(x)$  exists. Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Next we prove that  $\{f_n\}$  converges to  $f$  uniformly. Since

$\forall \epsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n, m \geq N \forall x \in D \quad |f_n(x) - f_m(x)| < \epsilon/2$ ,  
we have

$$|f_n(x) - f(x)| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \epsilon/2 < \epsilon.$$

Conversely, if  $\{f_n\}$  converges to  $f$  uniformly on  $D$ , then

$\forall \epsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n \geq N \forall x \in D \quad |f_n(x) - f(x)| < \epsilon/2$ .

Hence, by triangle inequality,

$\forall n, m \geq N \forall x \in D \quad |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon$ .

**$L^\infty$  norm.**

Consider a space of bounded function on a set  $D \subseteq \mathbb{R}$

$$L^\infty(D) = \{ f : D \rightarrow \mathbb{R} \text{ such that } \sup_{x \in D} |f(x)| < \infty \}$$

equipped with the  $L^\infty$  norm

$$\|f\|_\infty = \sup_{x \in D} |f(x)| < \infty.$$

Then, a sequence of functions  $\{f_n\}$  in  $L^\infty(D)$  converges uniformly  $f \in L^\infty(D)$  if and only if  $\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0$ ,

$$\text{i.e., } \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \|f_n - f\|_\infty \leq \epsilon.$$

Analogously, sequence  $\{f_n\}$  is Cauchy if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N \quad \|f_n - f_m\|_\infty \leq \epsilon.$$

### Continuous limit theorem.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Continuous Limit Theorem.** Suppose sequence  $\{f_n\}$  converges uniformly to  $f$  on  $D$ . Then, the following holds for any given  $x_0 \in D$ :

if  $f_n$  are continuous at  $x_0$ , then  $f$  is continuous at  $x_0$ .

**Proof:** Since

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \forall x \in D \quad |f_n(x) - f(x)| < \epsilon/3,$$

$$|f_N(x) - f(x)| < \epsilon/3$$

Also,  $\exists \delta > 0$  such that  $|f_N(x) - f_N(x_0)| < \epsilon/3$  whenever  $|x - x_0| < \delta$ .

Hence, by triangle inequality,

$$|f(x) - f(x_0)| \leq |f_N(x) - f(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \epsilon$$

whenever  $|x - x_0| < \delta$ .

## Differentiable limit theorem.

Consider a sequence of **differentiable** functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , on a closed interval  $D = [a, b] \subset \mathbb{R}$ .

**Differentiable Limit Theorem.** Suppose sequence of functions  $\{f_n\}$  converges pointwise to  $f$  and sequence  $\{f'_n\}$  converges uniformly to  $g$ . Then, function  $f$  is differentiable on  $[a, b]$  and  $f' = g$ .

**Proof:** For a given  $x_0 \in [a, b]$ , triangle inequality yields

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| + \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f'_n(x_0) \right| + |f'_n(x_0) - g(x_0)|$$

For a given  $\epsilon > 0$ , we have

(i) By uniform convergence,  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N \forall x \in [a, b] |f'_n(x) - g(x)| < \epsilon/3$ , and therefore,

$$\forall n \geq N |f'_n(x_0) - g(x_0)| < \epsilon/3.$$

(ii) By Cauchy Criterion,  $\exists N_0 \in \mathbb{N}$  s.t.  $\forall n, m \geq N_0 \forall x \in [a, b] |f'_n(x) - f'_m(x)| < \epsilon/3$

## Differentiable limit theorem.

**Proof (cont.):** For a given  $\epsilon > 0$ , we have

(i) By uniform convergence,  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N \forall x \in [a, b] |f'_n(x) - g(x)| < \epsilon/3$ ,  
and therefore,  $\forall n \geq N |f'_n(x_0) - g(x_0)| < \epsilon/3$ .

(ii) By Cauchy Criterion,  $\exists N_0 \in \mathbb{N}$  s.t.  $\forall n, m \geq N_0 \forall x \in [a, b] |f'_n(x) - f'_m(x)| < \epsilon/3$

Fix  $n \geq N + N_0$ . Then, by Mean Value Theorem,  $\exists c$  between  $x$  and  $x_0$  such that

$$\left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| = \left| \frac{f_m(x) - f_n(x)}{x - x_0} - \frac{f_m(x_0) - f_n(x_0)}{x - x_0} \right| = |f'_m(c) - f'_n(c)| < \epsilon/3$$

and taking  $m \rightarrow \infty$ , pointwise convergence of  $\{f_n\}$  implies

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| \leq \epsilon/3$$

(iii) By differentiability,  $\exists \delta > 0$  s.t.  $|x - x_0| < \delta \Rightarrow \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f'_n(x_0) \right| < \epsilon/3$ .

Hence,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

## Arzela-Ascoli theorem.

**Definition.** A collection of functions  $\mathcal{F}$  on a set  $D \subseteq \mathbb{R}$  is said to be **equicontinuous** if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall f \in \mathcal{F} \quad |x - y| < \delta \text{ in } D \Rightarrow |f(x) - f(y)| < \epsilon$$

**Arzela-Ascoli Theorem.** Consider a sequence of **equicontinuous** functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , on a closed interval  $D = [a, b] \subset \mathbb{R}$ . If functions  $\{f_n\}$  are **bounded** (i.e.,  $\exists M > 0$  s.t.  $\forall n \in \mathbb{N} \forall x \in D$   $|f_n(x)| \leq M$ ). Then, sequence  $\{f_n\}$  contains a uniformly convergent subsequence.

Arzela-Ascoli theorem can be restated in terms of space of bounded functions  $L^\infty(D)$  equipped with the  **$L^\infty$  norm**  $\|f\|_\infty = \sup_{x \in D} |f(x)|$ .

We say that a collection of functions  $\mathcal{F}$  is bounded if  $\exists M > 0$  such that  $\forall f \in \mathcal{F} \forall x \in D \quad |f(x)| \leq M$ .

**Arzela-Ascoli Theorem.** Consider a collection of **equicontinuous** functions  $\mathcal{F}$  on a closed interval  $D = [a, b] \subset \mathbb{R}$ . If  $\mathcal{F}$  is **closed** and **bounded**, then  $\mathcal{F}$  is **compact** in  $L^\infty[a, b]$ .

## Series of functions.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set  $D \subseteq \mathbb{R}$ .

**Definition.** The series of functions  $\sum_{k=1}^{\infty} f_k$  converges pointwise

to a function  $s$  on  $D$  if the sequence of its partial sums  $s_n = \sum_{k=1}^n f_k$  converges pointwise to  $s$  on  $D$ .

If  $\sum_{k=1}^{\infty} f_k$  converges pointwise to a function  $s$ , we write  $s = \sum_{k=1}^{\infty} f_k$ .

**Definition.** The series of functions  $\sum_{k=1}^{\infty} f_k$  converges uniformly

to a function  $s$  on  $D$  if the sequence of its partial sums  $s_n = \sum_{k=1}^n f_k$  converges uniformly to  $s$  on  $D$ .

## Series of functions.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a set

$D \subseteq \mathbb{R}$ . Let  $s_n = \sum_{k=1}^n f_k$  denote the corresponding sequence of

partial sums for the series of functions  $\sum_{k=1}^{\infty} f_k$ . Recall

**Continuous Limit Theorem.** Suppose sequence  $\{f_n\}$  converges uniformly to  $f$  on  $D$ . Then, the following holds for any given  $x_0 \in D$ :

if  $f_n$  are continuous at  $x_0$ , then  $f$  is continuous at  $x_0$ .

Applying the above Continuous Limit Theorem to the sequence of functions  $s_n$  yields the following result.

**Theorem.** Suppose series of functions  $\sum_{k=1}^{\infty} f_k$  converges uniformly to  $s$  on  $D$ . Then, the following holds for any given  $x_0 \in D$ :

if  $f_n$  are continuous at  $x_0$ , then  $s = \sum_{k=1}^{\infty} f_k$  is continuous at  $x_0$ .

## Series of functions.

Consider a sequence of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , defined on a closed interval  $D = [a, b] \subset \mathbb{R}$ . Let  $s_n = \sum_{k=1}^n f_k$  denote the corresponding sequence of **partial sums** for the series of functions  $\sum_{k=1}^{\infty} f_k$ . Recall

**Differentiable Limit Theorem.** Suppose sequence of functions  $\{f_n\}$  converges pointwise to  $f$  and sequence  $\{f'_n\}$  converges uniformly to  $g$ . Then, function  $f$  is differentiable on  $[a, b]$  and  $f' = g$ .

Applying the above Differentiable Limit Theorem to the sequence of functions  $s_n$  yields the following result.

**Theorem.** Suppose series of functions  $\sum_{k=1}^{\infty} f_k$  converges pointwise to  $s$  and series  $\sum_{k=1}^{\infty} f'_k$  converges uniformly to  $h$  on  $D = [a, b]$ .

Then,  $s = \sum_{k=1}^{\infty} f_k$  is differentiable on  $[a, b]$  and  $s' = h$ , i.e.,

$$\left( \sum_{k=1}^{\infty} f_k(x) \right)' = \sum_{k=1}^{\infty} f'_k(x) \quad \forall x \in [a, b].$$