MTH 306 - Lecture 22

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Topics:

- Linear systems in matrix form.
- Matrix products.
- The inverse of a square matrix.

MATRIX $\times$ VECTOR product.

Consider an $n \times m$ matrix $A = \begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ & \cdots & \\ - & \vec{r}_n & - \end{bmatrix}$

and an m-**vector** $\vec{x}$.

The matrix $\times$ vector product is defined as

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \cdots \\ \vec{r}_n \cdot \vec{x} \end{bmatrix}$$

- a column vector of dot products of row vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ with $\vec{x}$

MATRIX $\times$ VECTOR product.

An $n \times m$ matrix $A = \begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ & \cdots & \\ - & \vec{r}_n & - \end{bmatrix}$ can be multiplied only by m-**vectors** since its row vectors

$$\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$$

are all m-vectors.

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \cdots \\ \vec{r}_n \cdot \vec{x} \end{bmatrix}$$

MATRIX × VECTOR product.

- **Example.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Then

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 2 \times 2 + 3 \times (-1) \\ 2 \times 3 + (-1) \times 2 + 3 \times (-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

• **Example.** Given a matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and a vector $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Find $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$A\vec{x} = \vec{b}$$

Solution: $A\vec{x} = \vec{b}$ can be rewritten as

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Multiplying out, obtain

$$\begin{bmatrix} x + y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Leftrightarrow \begin{cases} x + y = 1 \\ 3x + 4y = 5 \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 2 \end{cases}$$

Linear systems in matrix form.

$$A\vec{x} = \vec{b} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Augmented form: $[A|\vec{b}]$ denotes $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$

In general, solving equation $A\vec{x} = \vec{b}$ with matrix A and vector $\vec{b}$ both known, and vector $\vec{x}$ unknown is **equivalent** to solving a system of linear equations $[A|\vec{b}]$ (in augmented form).

BIG PICTURE: the inverse of a square matrix.

• 1-D case: solve $Ax = b$

Solution: if $A \neq 0$ find A^{-1}, and multiply by b:

$$x = A^{-1}b$$

• Dimension ≥ 2 case: solve $A\vec{x} = \vec{b}$, where A is $n \times n$ square matrix, and $\vec{b}$ is an n-vector.

Solution: if det(A) $\neq 0$ find A^{-1}, and multiply by $\vec{b}$:

$$\vec{x} = A^{-1}\vec{b}$$

MATRIX $\times$ VECTOR product.

The properties of dot product carry over to matrix $\times$ vector products:

- $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

- $A(c\vec{v}) = c(A\vec{v})$, for any constant c

Given a 2×2 matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and a 2-vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $I\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}$.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2×2 **identity matrix.**

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 **identity matrix.**

Identity matrix.

In general,
$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is an $n \times n$ **identity matrix.**

- If $\vec{x}$ is an n-vector, then $I\vec{x} = \vec{x}$.
- If A is an $n \times n$ matrix, then $AI = A = IA$.

MATRIX $\times$ MATRIX product.

Consider two matrices,

$$A = \begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ & \cdots & \\ - & \vec{r}_n & - \end{bmatrix} \text{ and } B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix}$$

$n \times m$ matrix and $m \times k$ matrix

Then, the product of A and B is defined as

$$AB = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{b}_1 & \vec{r}_1 \cdot \vec{b}_2 & \cdots & \vec{r}_1 \cdot \vec{b}_k \\ \vec{r}_2 \cdot \vec{b}_1 & \vec{r}_2 \cdot \vec{b}_2 & \ldots & \vec{r}_2 \cdot \vec{b}_k \\ \cdots & \cdots & \cdots & \cdots \\ \vec{r}_n \cdot \vec{b}_1 & \vec{r}_n \cdot \vec{b}_2 & \cdots & \vec{r}_n \cdot \vec{b}_k \end{bmatrix}$$

- an $n \times k$ matrix.

Example.

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix}$$

$$AB = \begin{bmatrix} | & | \\ A\vec{b}_1 & A\vec{b}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{b}_1 & \vec{r}_1 \cdot \vec{b}_2 \\ \vec{r}_2 \cdot \vec{b}_1 & \vec{r}_2 \cdot \vec{b}_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (-1) \times 2 + 1 \times 1 & (-1) \times 1 + 1 \times (-2) \\ 1 \times 2 + 1 \times 1 & 1 \times 1 + 1 \times (-2) \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$$

$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

We have shown:

$$AB = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$$

However, $BA = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$

In general, $\mathbf{AB} \neq \mathbf{BA}$ (in the sense AB may **or** may not be equal to BA).

Example. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 1 \end{bmatrix}$

- $AB = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 & 9 \\ -4 & -2 & 3 & -1 \\ 0 & 4 & 10 & 10 \end{bmatrix}$

- **NO** $BA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 2 \end{bmatrix}$

MATRIX $\times$ MATRIX product.

Rules of matrix multiplication:

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $c(AB) = (cA)B = A(cB)$

for any constant c

Linear systems in matrix form.

$$A\vec{x}=\vec{b} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} ax+by=e \\ cx+dy=f \end{cases}$$

Augmented form: $[A|\vec{b}]$ denotes $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$

In general, solving equation $A\vec{x}=\vec{b}$ with matrix A and vector $\vec{b}$ both known, and vector $\vec{x}$ unknown is **equivalent** to solving a system of linear equations $[A|\vec{b}]$ (in augmented form).

BIG PICTURE: the inverse of a square matrix.

- 1-D case: solve $Ax = b$

Solution: if $A \neq 0$ find A^{-1}, and multiply by b:

$$x = A^{-1}b$$

- Dimension ≥ 2 case: solve $A\vec{x} = \vec{b}$, where A is $n \times n$ square matrix, and $\vec{b}$ is an n-vector.

Solution: if det$(A) \neq 0$ find A^{-1}, and multiply by $\vec{b}$:

$$\vec{x} = A^{-1}\vec{b}$$

Matrix inverse A^{-1}.

Given a square matrix A such that

$$\det(A) \neq 0,$$

i.e. $A\vec{x} = \vec{b}$ has a unique solution.

The inverse A^{-1} of A is a square matrix such that

$$A^{-1}A = I = AA^{-1}$$

Once A^{-1}, the unique solution is obtained:

$$\vec{x} = A^{-1}\vec{b}$$

Matrix inverse A^{-1}.

$$A\vec{x} = \vec{b} \quad \Leftrightarrow \quad A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$\Leftrightarrow \quad (A^{-1}A)\vec{x} = A^{-1}\vec{b} \quad \Leftrightarrow \quad I\vec{x} = A^{-1}\vec{b}$$

$$\Leftrightarrow \quad \vec{x} = A^{-1}\vec{b}$$

So,

$$A\vec{x} = \vec{b} \quad \Leftrightarrow \quad \vec{x} = A^{-1}\vec{b}$$

Finding A^{-1}

Row elimination: $A\vec{x} = I\vec{b} \;\rightarrow\; I\vec{x} = A^{-1}\vec{b}$

In augmented form: $[A|I] \;\rightarrow\; [I|A^{-1}]$

- **Example.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^{-1} =?$

System $A\vec{x} = I\vec{b}$	Augmented form $[A \mid I]$
$\begin{cases} x + 2y = b_1 \\ 3x + 4y = b_2 \end{cases}$	$\left[\begin{array}{cc\|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$ $\begin{array}{c} R1 \\ R2 \end{array}$
$\Downarrow$	$\Downarrow$
$\begin{cases} x + 2y = b_1 \\ -2y = -3b_1 + b_2 \end{cases}$	$\left[\begin{array}{cc\|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$ $\begin{array}{c} R1 \\ R2 - 3R1 \end{array}$
$\Downarrow$	$\Downarrow$
$\begin{cases} x + 2y = b_1 \\ y = \frac{3}{2}b_1 - \frac{1}{2}b_2 \end{cases}$	$\left[\begin{array}{cc\|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$ $\begin{array}{c} R1 \\ -\frac{1}{2}R2 \end{array}$
$\Downarrow$	$\Downarrow$
$\begin{cases} x = -2b_1 + b_2 \\ y = \frac{3}{2}b_1 - \frac{1}{2}b_2 \end{cases}$	$\left[\begin{array}{cc\|cc} 1 & 0 & \mathbf{-2} & \mathbf{1} \\ 0 & 1 & \mathbf{\frac{3}{2}} & \mathbf{-\frac{1}{2}} \end{array}\right]$ $\begin{array}{c} R1 - 2R2 \\ R2 \end{array}$
$\vec{x} = A^{-1}\vec{b}$	$[I \mid A^{-1}]$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ Our answer: } A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Check:

- $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- $A^{-1}A = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$