MTH 306 - Lecture 10

Yevgeniy Kovchegov
Oregon State University

Topics:

- The root test.
- The ratio test.

The root test.

Let $a_k \geq 0$. Then

$(a_k)^{1/k} \leq r$ for all large k and some $r<1$ $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges

$(a_k)^{1/k} \geq r$ for all large k and some $r\geq 1$ $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ diverges

Proof: $(a_k)^{1/k} \leq r \quad \Rightarrow \quad a_k \leq r^k$.

Thus if $r < 1$, $\sum_{k=1}^{\infty} a_k$ converges by the basic comparison test.

Similarly, $(a_k)^{1/k} \geq r \quad \Rightarrow \quad a_k \geq r^k \geq 1$ if $r \geq 1$.

The root test.

Let $a_k \geq 0$. Then

$$(a_k)^{1/k} \leq r \quad \text{for all large } k \quad \text{and some } r<1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{converges}$$

$$(a_k)^{1/k} \geq r \quad \text{for all large } k \quad \text{and some } r\geq 1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{diverges}$$

Example: $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k^2+1}{3k^2+k+5}\right)^k$

Solution: $(a_k)^{1/k} = \frac{2k^2+1}{3k^2+k+5} \to \frac{2}{3}$ as $k \to \infty$.

Thus for k large enough, $(a_k)^{1/k} = \frac{2k^2+1}{3k^2+k+5} \leq r = \frac{3}{4} < 1$.

Hence the series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k^2+1}{3k^2+k+5}\right)^k$ converges.

Example. Determine whether the series

$$\sum_{k=1}^{\infty}\left(\frac{(2+\sin k)\cdot k^2}{5k^2-3}\right)^k$$

is convergent or divergent.

Solution: $-1 \le \sin x \le 1 \Rightarrow 1 \le (2+\sin x) \le 3$

Therefore $(a_k)^{1/k} = \frac{(2+\sin k)\cdot k^2}{5k^2-3} \le \frac{3k^2}{5k^2-3} \le r = \frac{4}{5}$

for k large enough as

$$\lim_{k\to\infty}\frac{3k^2}{5k^2-3} = \lim_{k\to\infty}\frac{3}{5-3/k^2} = \frac{3}{5} < \frac{4}{5}$$

So, $(a_k)^{1/k} \le r = \frac{4}{5} < 1$ and the series converges.

Example. Determine whether the series

$$\sum_{k=1}^{\infty}\left(\frac{6k\sqrt{k}+70k+55}{5k\sqrt{k}+700k+555}\right)^k$$

is convergent or divergent.

Solution:

$$(a_k)^{1/k} = \frac{6k\sqrt{k}+70k+55}{5k\sqrt{k}+700k+555} = \frac{6+70/\sqrt{k}+55/(k\sqrt{k})}{5+700/\sqrt{k}+555/(k\sqrt{k})} \longrightarrow \frac{6}{5}$$

as $k \to \infty$. Thus for k large enough,

$$(a_k)^{1/k} = \frac{6k\sqrt{k}+70k+55}{5k\sqrt{k}+700k+555} \geq r=1$$

Hence the series $\sum_{k=1}^{\infty}\left(\frac{6k\sqrt{k}+70k+55}{5k\sqrt{k}+700k+555}\right)^k$ diverges.

The root test.

In general, if $|a_k|^{1/k} \to L$ (finite or infinite) as $k \to \infty$, then

- $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $L < 1$
- $\sum_{k=1}^{\infty} a_k$ **diverges** if $L > 1$
- **unable to determine** using this test if $L = 1$

Example. Determine whether the series

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution: Recall that $\lim\limits_{x\to\infty}(1+\frac{a}{x})^x = e^a$.

Therefore

$$|a_n|^{1/n} = \left(1-\frac{1}{n}\right)^n \to L=e^{-1}=\frac{1}{e}<1$$

as $n \to \infty$.

Thus $\sum\limits_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges absolutely.

The ratio test.

Let $a_k > 0$. Then

$$\frac{a_{k+1}}{a_k} \le r \quad \text{for all large } k \quad \text{and some } r<1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{converges}$$

$$\frac{a_{k+1}}{a_k} \ge r \quad \text{for all large } k \quad \text{and some } r\ge 1 \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{diverges}$$

Example. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^2}{k!}$$

is convergent or divergent.

Solution: $a_k = \frac{k^2}{k!} > 0$ and

$$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{(k+1)^2}{(k+1)!}\right)}{\left(\frac{k^2}{k!}\right)} = \left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{k+1} = \frac{k+1}{k^2} \to 0$$

as $k \to \infty$.

Thus $\frac{a_{k+1}}{a_k} \leq r = \frac{1}{2} < 1$ for k large enough. Hence the series converges.

The ratio test.

In general, if $\left|\frac{a_{k+1}}{a_k}\right| \to L$ (finite or infinite) as $k \to \infty$, then

- $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $L < 1$
- $\sum_{k=1}^{\infty} a_k$ **diverges** if $L > 1$
- **unable to determine** using this test if $L = 1$

Example. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

is convergent or divergent.

Solution: $a_k = \frac{k^k}{k!} > 0$ and

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\left(\frac{(k+1)^{k+1}}{(k+1)!}\right)}{\left(\frac{k^k}{k!}\right)} = \frac{(k+1)^{k+1}}{k^k} \cdot \frac{1}{k+1} = \left(\frac{k+1}{k}\right)^k = \left(1+\frac{1}{k}\right)^k \to L=e>1$$

as $k \to \infty$. Thus $\left|\frac{a_{k+1}}{a_k}\right| \to L=e > 1$. Hence the series diverges.

Alternatively, we could notice that $\frac{k^k}{k!} = \frac{k \cdot k \cdot \ldots \cdot k}{1 \cdot 2 \cdot \ldots \cdot k} \geq 1$ and use the divergence test.

Example. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k!}{k^k}$$

is convergent or divergent.

Solution: $a_k = \frac{k!}{k^k} > 0$ and

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\left(\frac{(k+1)!}{(k+1)^{k+1}}\right)}{\left(\frac{k!}{k^k}\right)} = \frac{(k+1)\cdot k^k}{(k+1)^{k+1}} = \left(\frac{k}{k+1}\right)^k = \frac{1}{\left(1+\frac{1}{k}\right)^k} \to L = \frac{1}{e} < 1$$

as $k \to \infty$. Thus $\left|\frac{a_{k+1}}{a_k}\right| \to L = \frac{1}{e} < 1$.

Hence the series converges.