Figure: Euclid of Alexandria was a Greek mathematician, often referred to as the “Father of Geometry”
6.2: Right triangle trigonometry

Standard labeling for any triangle

Vertices: $A$, $B$ and $C$
Angles: $\alpha$, $\beta$ and $\gamma$
Sides: $a$, $b$ and $c$

$a$ is opposite to $\alpha$
$b$ is opposite to $\beta$
$c$ is opposite to $\gamma$
Similar triangles

- Similar triangles have congruent corresponding angles.
- Corresponding sides of similar triangles are proportional.
- They are NOT necessarily the same size.
Six possible side ratios:

\[
\frac{\text{opp}}{\text{hyp}}, \frac{\text{adj}}{\text{hyp}}, \frac{\text{opp}}{\text{adj}}, \frac{\text{hyp}}{\text{opp}}, \frac{\text{hyp}}{\text{adj}}, \frac{\text{adj}}{\text{opp}}
\]
Right triangle definitions of trigonometric functions

Let $\theta$ be the acute angle of a right triangle.

\[
\begin{align*}
\sin(\theta) &= \frac{\text{opp}}{\text{hyp}}, & \cos(\theta) &= \frac{\text{adj}}{\text{hyp}}, & \tan(\theta) &= \frac{\text{opp}}{\text{adj}}, \\
\csc(\theta) &= \frac{\text{hyp}}{\text{opp}}, & \sec(\theta) &= \frac{\text{hyp}}{\text{adj}}, & \cot(\theta) &= \frac{\text{adj}}{\text{opp}}
\end{align*}
\]
Example

Example 1

Find the six trigonometric functions for the angle $\theta$, where the length of the opposite side is 5 and the length of the hypotenuse is 13.

Solution:
By the Pythagorean theorem, the length of the side adjacent to $\theta$ is:

$$\text{adjacent} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12.$$

Therefore, we have:

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13}, \quad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13}, \quad \tan(\theta) = \frac{\text{opp}}{\text{adj}} = ?,$$

$$\csc(\theta) = \frac{\text{hyp}}{\text{opp}} = ?, \quad \sec(\theta) = \frac{\text{hyp}}{\text{adj}} = ?, \quad \cot(\theta) = \frac{\text{adj}}{\text{opp}} = ?$$
Reciprocal identities (or ratio identities)

\[
csc(\theta) = \frac{1}{\sin(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \cot(\theta) = \frac{1}{\tan(\theta)}.
\]

**Example 2**

Find the exact value of: \(\cot(45^\circ)\), \(\sec(30^\circ)\) and \(\csc(60^\circ)\).
Cofunction identities

Write the values of each trigonometric function for the angles $\alpha$ and $\beta$:

\[
\sin(\alpha) = \frac{a}{c} = \cos(\beta)
\]
\[
\tan(\alpha) = \frac{a}{b} = \cot(\beta)
\]
\[
\sec(\alpha) = \frac{c}{b} = \csc(\beta).
\]
### Some exact values of trigonometric functions

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
<th>Trigonometry Function Table</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>sin(θ)</td>
</tr>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>π/6</td>
<td>1/2</td>
</tr>
<tr>
<td>45°</td>
<td>π/4</td>
<td>√2/2</td>
</tr>
<tr>
<td>60°</td>
<td>π/3</td>
<td>√3/2</td>
</tr>
<tr>
<td>90°</td>
<td>π/2</td>
<td>1</td>
</tr>
</tbody>
</table>
\( \alpha \) and \( \beta \) are complementary angles

\[
\alpha + \beta = 90^\circ 
\]

\[
\sin(\alpha) = \cos(90^\circ - \alpha) = \cos(\beta), \quad \cos(\alpha) = \sin(90^\circ - \alpha) = \sin(\beta),
\]
\[
\tan(\alpha) = \cot(90^\circ - \alpha) = \cot(\beta), \quad \cot(\alpha) = \tan(90^\circ - \alpha) = \tan(\beta),
\]
\[
\sec(\alpha) = \csc(90^\circ - \alpha) = \csc(\beta), \quad \csc(\alpha) = \sec(90^\circ - \alpha) = \sec(\beta).
\]

Cofunctions of complementary angles are equal.
Example 3

It is known that \( \tan \left( \frac{\pi}{8} \right) = \sqrt{2} - 1 \). Find \( \tan \left( \frac{3\pi}{8} \right) \).

\[
\tan \left( \frac{3\pi}{8} \right) = \tan \left( \frac{\pi}{2} - \frac{\pi}{8} \right) \\
= \cot \left( \frac{\pi}{8} \right) \\
= \frac{1}{\tan \left( \frac{\pi}{8} \right)} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1
\]
Angle of elevation and angle of depression

Many times in application examples involving trigonometry, we consider angles formed by a horizontal line and the line of sight from a reference point on the horizontal line to an object below or above it. We refer to such an angle as:

- **Angle of elevation** if the object is above the horizontal.
- **Angle of depression** if the object is below the horizontal.
Example 4: Height of the world tallest tree

According to the *Guinness Book of World Records*, the tallest tree currently standing is the Mendocino Tree, a coast redwood at Montgomery State Reserve near Ukiah, California. At a distance of 200 feet from the base of the tree, the angle of elevation to the top of the tree is approximately $61.5^\circ$. How tall is the tree?
\[
\tan(61.5^\circ) = \frac{h}{200}
\]

\[
h = 200 \tan(61.5^\circ)
\]

\[
= 200 \times 1.8417 \approx 368.35
\]
Example

**Example 5: Pinpointing a Fire**

A U.S. Forest Service helicopter is flying at a height of 500 feet. The pilot spots a fire in the distance with an angle of depression of 12°. Find the horizontal distance to the fire.
Example

\[
\tan(12^\circ) = \frac{500}{d}
\]

\[
d = \frac{500}{\tan(12^\circ)}
\]

\[
= \frac{500}{0.2125} \approx 2352.315
\]