

shallow minimum for some angle θ_1 , whose value depends on the optical constants. This angle is called the *principal angle of incidence* and corresponds to the Brewster angle for dielectrics.

If linearly polarized light, whose polarization is neither pure *TE* nor pure *TM*, is reflected from a metal, then the reflected light will be elliptically polarized, in general. The intensity and the polarization of the reflected light can be calculated from the above theory. This involves knowing the complex index of refraction \mathcal{N} . Conversely, it is possible to determine \mathcal{N} by appropriate measurements of the intensity and polarization of the reflected light. The method is known as *ellipsometry*. For more information the reader is referred to Reference [5].

Normal Incidence In the case of normal incidence, both Equations (6.80) and (6.81) reduce to the same result, namely,

$$r_s = r_p = \frac{1 - \mathcal{N}}{1 + \mathcal{N}} = \frac{1 - n - ik}{1 + n + ik} \quad (6.82)$$

The following expression for the normal reflectance is then obtained:

$$R = \left| \frac{1 - \mathcal{N}}{1 + \mathcal{N}} \right|^2 = \frac{(1 - n)^2 + \kappa^2}{(1 + n)^2 + \kappa^2} \quad (6.83)$$

This reduces to the previously found value for dielectrics (Section 2.7) as κ approaches zero and the index of refraction becomes real. On the other hand, for metals the extinction coefficient κ is large. This results in a high value of the reflectance which approaches unity as κ becomes infinite.

In the previous section we showed that for metals, both n and κ in fact become very large and approach the value $\sqrt{\sigma/2\omega\epsilon_0}$ in the limit of low frequencies, Equation (6.51). It is easy to show from Equation (6.83) that the reflectance in this case is given by the approximate formula

$$R \approx 1 - \frac{2}{n} \approx 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}} \quad (6.84)$$

This is known as the *Hagen-Rubens formula*. It has been verified experimentally for a number of metals in the far infrared. The formula predicts that for a given metal the quantity $(1 - R)^2$ is proportional to the frequency or inversely proportional to the wavelength. All of the good conductors—copper, silver, gold, and so on—are excellent reflectors in the near infrared region ($\lambda \approx 1$ to 2μ) and are even better reflectors in the far infrared ($\lambda > 20 \mu$), where the reflectance becomes very nearly unity.

6.7 Propagation of Light in Crystals

The distinguishing basic feature of the crystalline state, as far as optical properties are concerned, is the fact that crystals are generally electrically anisotropic. This means that the polarization produced in the crystal by a given electric field is not just a simple scalar constant times the field, but varies in a manner that depends on the direction of the applied field in relation to the crystal lattice. One of the consequences is that the speed of propagation of a light wave in a crystal is a function of the direction of propagation and the polarization of the light.

It turns out that there are generally two possible values of the phase velocity for a given direction of propagation. These two values are associated with mutually orthogonal polarizations of the light waves. Crystals are said to be *doubly refracting* or *birefringent*. Actually not all crystals exhibit double refraction. Whether they do or do not depends on their symmetry. Crystals of the cubic class of symmetry, such as sodium chloride, never exhibit double refraction, but are optically isotropic. All crystals, other than cubic crystals, do show double refraction, however.

A model to illustrate the anisotropic polarizability of a crystal is shown in Figure 6.7. A bound electron is pictured here as attached to

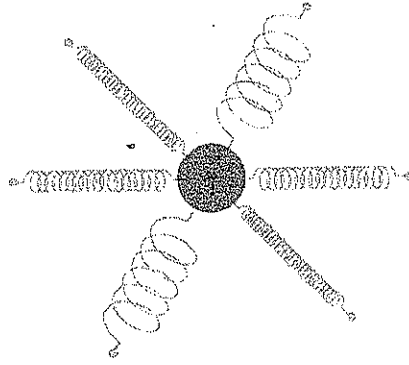


Figure 6.7. Model to show anisotropic binding of an electron in a crystal.

a set of fictitious elastic springs. The springs have different stiffnesses for different directions of the electron's displacement from its equilibrium position within the crystal lattice. Consequently, the dis-

placement of the electron under the action of an external field \mathbf{E} depends on the direction of the field as well as its magnitude. This is also true of the resulting polarization \mathbf{P} .

The dependence of \mathbf{P} on \mathbf{E} is expressible as a tensor relation in the form

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad (6.85)$$

This is customarily abbreviated as

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \quad (6.86)$$

where χ is the susceptibility tensor:

$$\chi = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \quad (6.87)$$

The corresponding displacement vector \mathbf{D} is given by $\mathbf{D} =$

$$\epsilon_0(\mathbf{1} + \chi)\mathbf{E} = \epsilon\mathbf{E}, \text{ where } \mathbf{1} \text{ is the unit matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and} \quad (6.88)$$

which is known as the *dielectric tensor*.

For ordinary nonabsorbing crystals the χ tensor is symmetric so there always exists a set of coordinate axes, called *principal axes*, such that the χ tensor assumes the diagonal form

$$\chi = \begin{bmatrix} \chi_{11} & 0 & 0 \\ 0 & \chi_{22} & 0 \\ 0 & 0 & \chi_{33} \end{bmatrix} \quad (6.89)$$

The three χ s are known as the *principal susceptibilities*. Corresponding to these, the quantities $K_{11} = 1 + \chi_{11}$, . . . , and so forth, are called the principal *dielectric constants*.

In view of Equation (6.86), the general wave equation (6.14) can be written in the following form:

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{c^2} \chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (6.90)$$

It then follows that the crystal can sustain monochromatic plane waves of the usual form $e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ provided the propagation vector \mathbf{k} satisfies the equation

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{\omega^2}{c^2} \mathbf{E} = -\frac{\omega^2}{c^2} \chi \mathbf{E} \quad (6.91)$$

Written out in terms of components, the above equation is equivalent to the following three equations:

$$\begin{aligned} (-k_y^2 - k_z^2 + \frac{\omega^2}{c^2}) E_x + k_x k_y E_y + k_x k_z E_z &= -\frac{\omega^2}{c^2} \chi_{11} E_x \\ k_y k_x E_x + (-k_x^2 - k_z^2 + \frac{\omega^2}{c^2}) E_y + k_y k_z E_z &= -\frac{\omega^2}{c^2} \chi_{22} E_y \\ k_z k_x E_x + k_z k_y E_y + (-k_x^2 - k_y^2 + \frac{\omega^2}{c^2}) E_z &= -\frac{\omega^2}{c^2} \chi_{33} E_z \end{aligned} \quad (6.92)$$

In order to interpret the physical meaning of these equations, suppose we have a particular case of a wave propagating in the direction of one of the principal axes, say the x axis. In this case $k_x = k$, $k_y = k_z = 0$, and the three equations reduce to

$$\begin{aligned} \frac{\omega^2}{c^2} E_x &= -\frac{\omega^2}{c^2} \chi_{11} E_x \\ (-k^2 + \frac{\omega^2}{c^2}) E_y &= -\frac{\omega^2}{c^2} \chi_{22} E_y \\ (-k^2 + \frac{\omega^2}{c^2}) E_z &= -\frac{\omega^2}{c^2} \chi_{33} E_z \end{aligned} \quad (6.93)$$

The first equation implies that $E_x = 0$, because neither ω nor χ_{11} is zero. This means that the \mathbf{E} field is transverse to the x axis, which is the direction of propagation. Consider next the second equation. If $E_y \neq 0$, then

$$k = \frac{\omega}{c} \sqrt{1 + \chi_{22}} = \frac{\omega}{c} \sqrt{K_{22}} \quad (6.94)$$

The third equation, likewise, implies that if $E_z \neq 0$, then

$$k = \frac{\omega}{c} \sqrt{1 + \chi_{33}} = \frac{\omega}{c} \sqrt{K_{33}} \quad (6.95)$$

Now ω/k is the phase velocity of the wave. Thus we have two possible phase velocities, namely, $c/\sqrt{K_{22}}$ if the \mathbf{E} vector points in the y direction, and $c/\sqrt{K_{33}}$ if the \mathbf{E} vector is in the z direction.

More generally we can show that for any direction of the propagation vector \mathbf{k} , there are two possible values of the magnitude k and hence two possible values of the phase velocity. To do this, let us introduce the three *principal indices of refraction* n_1 , n_2 , and n_3 , defined by

$$\begin{aligned} n_1 &= \sqrt{1 + \chi_{11}} = \sqrt{K_{11}} \\ n_2 &= \sqrt{1 + \chi_{22}} = \sqrt{K_{22}} \\ n_3 &= \sqrt{1 + \chi_{33}} = \sqrt{K_{33}} \end{aligned} \quad (6.96)$$

equal to zero. Setting the first factor equal to zero gives the equation of a circle

$$k_x^2 + k_y^2 = \left(\frac{n_3\omega}{c}\right)^2 \tag{6.99}$$

The second factor gives the equation of an ellipse

$$\frac{k_x^2}{(n_2\omega/c)^2} + \frac{k_y^2}{(n_1\omega/c)^2} = 1 \tag{6.100}$$

Similar equations are obtained for the xz and the yz planes. The intercept of the \mathbf{k} surface with each coordinate plane therefore consists of one circle and one ellipse as shown. The complete \mathbf{k} surface is double; that is, it consists of an inner sheet and an outer sheet. This implies that for any given direction of the wave vector \mathbf{k} , there are two possible values for the wavenumber k . It follows that there are also two values of the phase velocity. Now we just showed that for a wave propagating in the x direction, the two phase velocities correspond to two mutually orthogonal directions of polarization. It turns out that the same is true for any direction of propagation; that is, the two phase velocities always correspond to two mutually orthogonal polarizations [5]. Now, as we know, a light wave of arbitrary polarization can always be resolved into two orthogonally polarized waves. Hence, when unpolarized light, or light of arbitrary polarization propagates through a crystal, it can be considered to consist of two independent waves that are polarized orthogonally with respect to each other and traveling with different phase velocities.

The nature of the \mathbf{k} surface is such, that the inner and outer sheets touch at a certain point P as shown in Figure 6.8. This point defines a direction for which the two values of k are equal. The direction so defined is called an *optic axis* of the crystal. Thus, when propagating in the direction of an optic axis, the phase velocities of the two orthogonally polarized waves reduce to the same value.

The general case is shown in Figures 6.8 and 6.9(a). Here the three principal indices n_1 , n_2 , and n_3 are all different. It is easy to see from the intercepts that there are *two* optic axes. In this case the crystal is said to be *biaxial*. In many crystals it happens that two of the principal indices are equal, in which case there is only *one* optic axis and the crystal is called *uniaxial*. The \mathbf{k} surface for a uniaxial crystal consists of a sphere and an ellipsoid of revolution, the axis of which is the optic axis of the crystal [Figure 6.9(b) and (c)]. If all three indices are equal, then the \mathbf{k} surface degenerates to a single sphere, and the crystal is not doubly refracting at all but is optically isotropic.

In view of the fact that the principal indices are related to the

Now in Equation (6.92), in order for a nontrivial solution for E_x , E_y , and E_z to exist, the determinant of the coefficients must vanish, namely,

$$\begin{vmatrix} (n_1\omega/c)^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & (n_2\omega/c)^2 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & (n_3\omega/c)^2 - k_x^2 - k_y^2 \end{vmatrix} = 0 \tag{6.97}$$

where we have used Equation (6.96). The above equation can be represented by a three-dimensional surface in \mathbf{k} space. The form of this \mathbf{k} surface, or wave-vector surface, is shown in Figure 6.8. To see

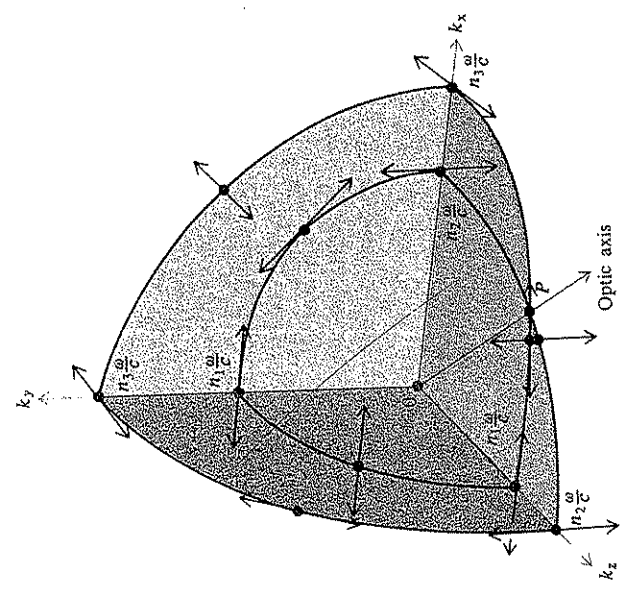


Figure 6.8. The wave-vector surface.

how the surface is constructed, consider any one of the coordinate planes, say the xy plane. In this plane $k_z = 0$, and the determinant reduces to the product of the two factors

$$\left[\left(\frac{n_3\omega}{c}\right)^2 - k_x^2 - k_y^2 \right] \left[\left(\frac{n_1\omega}{c}\right)^2 - k_x^2 \right] - k_x^2 k_y^2 = 0 \tag{6.98}$$

Since the product must vanish, either or both of the factors must be

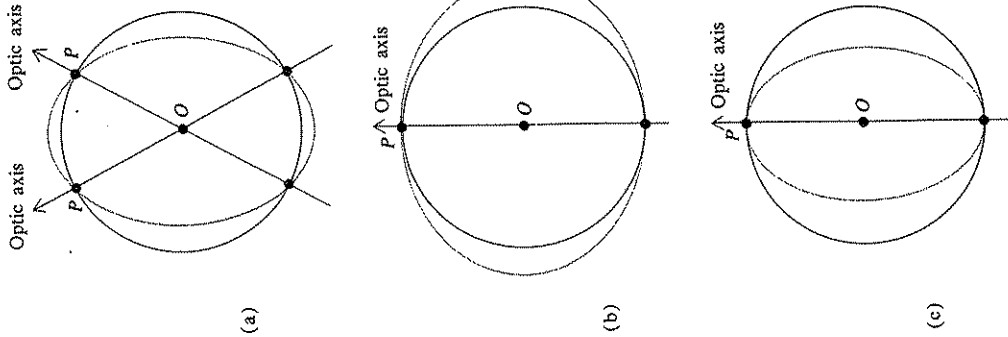


Figure 6.9. Intercepts of the wave-vector surfaces in the xz plane for (a) biaxial crystals; (b) uniaxial positive crystals; (c) uniaxial negative crystals.

components of the χ tensor by Equation (6.96), we can conveniently classify crystals according to the χ tensor as follows:

Isotropic cubic	$\chi = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	$\chi_{11} = \chi_{22} = \chi_{33} = a$	$n = \sqrt{1+a}$
Uniaxial trigonal tetragonal hexagonal	$\chi = \begin{bmatrix} a & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & b \end{bmatrix}$	$\chi_{11} = \chi_{22} = a, \chi_{33} = b$	$n_0 = \sqrt{1+a}$
			$n_E = \sqrt{1+b}$
Biaxial/ triclinic monoclinic orthorhombic	$\chi = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$	$\chi_{11} = a$	$\chi_{22} = b$
			$\chi_{33} = c$
		$n_1 = \sqrt{1+a}$	$n_2 = \sqrt{1+b}$
		$n_3 = \sqrt{1+c}$	

In a uniaxial crystal the index of refraction that corresponds to the two equal elements, $\chi_{11} = \chi_{22}$, is called the *ordinary* index n_0 , and the other index, corresponding to χ_{33} , is called the *extraordinary* index n_E . If $n_0 < n_E$, the crystal is said to be *positive*; whereas if $n_0 > n_E$, it is called a *negative* crystal. Table 6.1 lists some examples of crystals with their indices of refraction.

Phase-Velocity Surface² Knowing that the wavenumber k is related to the magnitude v of the phase velocity by $k = \omega/v$, we can write the relation vectorially as

$$\mathbf{k} = v \frac{\omega}{v^2} \tag{6.101}$$

In terms of components the above vector equation is equivalent to the three scalar equations

$$k_x = v_x \frac{\omega}{v^2} \quad k_y = v_y \frac{\omega}{v^2} \quad k_z = v_z \frac{\omega}{v^2} \tag{6.102}$$

Let us substitute the above values into the equation of the k surface in Equation (6.97). The result is

$$\left| \begin{array}{ccc} n_1^2 v^4 / c^2 - v_y^2 - v_z^2 & v_x v_y & v_x v_z \\ v_y v_x & n_2^2 v^4 / c^2 - v_x^2 - v_z^2 & v_y v_z \\ v_z v_x & v_z v_y & n_3^2 v^4 / c^2 - v_x^2 - v_y^2 \end{array} \right| = 0 \tag{6.103}$$

² In this section we depart from the notation used in the other parts of the book, namely, we use this letter v for phase velocity, rather than u , which will be used here for ray velocity.

Table 6.1. SOME COMMON CRYSTALS

OPTICALLY ISOTROPIC (CUBIC) CRYSTALS			
n			
Sodium chloride	1.544		
Diamond	2.417		
Fluorite	1.392		
UNIAXIAL POSITIVE CRYSTALS			
n_o	n_E		
Ice	1.309		
Quartz	1.544		
Zircon	1.923		
Rutile	2.616		
UNIAXIAL NEGATIVE CRYSTALS			
n_o	n_E		
Beryl	1.598		
Sodium nitrate	1.587		
Calcite	1.658		
Tourmaline	1.669		
BIAXIAL CRYSTALS			
n_1	n_2	n_3	
Gypsum	1.520	1.523	1.530
Feldspar	1.522	1.526	1.530
Mica	1.552	1.582	1.588
Topaz	1.619	1.620	1.627

after cancellation of ω^2 and division by v^4 . This equation defines a three-dimensional surface that can be considered as the *reciprocal surface* to the \mathbf{k} surface. It is called the *phase-velocity surface*. It is a double-sheeted surface and gives directly the two possible values of the phase velocity for a given direction of a plane wave propagating in the crystal. The general form of the phase-velocity surface is shown in Figure 6.10. The intercepts with the coordinate planes consists of circles and *fourth-degree ovals*. Thus for the xy plane the two equations for the intercepts are

$$v^2 = v_x^2 + v_y^2 = \frac{C^2}{n_3^2} \tag{6.104}$$

$$\frac{v_x^2}{n_2^2} + \frac{v_y^2}{n_1^2} = \frac{v^4}{C^2} \tag{6.105}$$

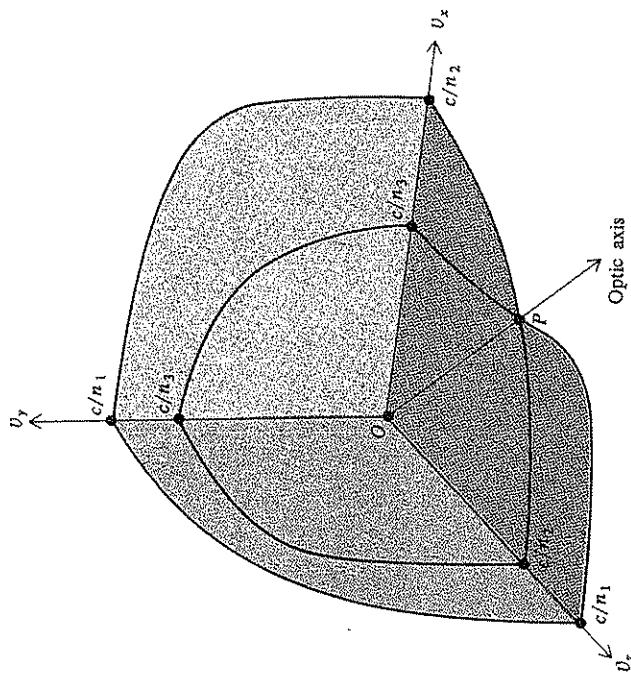


Figure 6.10. The phase-velocity surface.

Similar relations hold for the other coordinate planes.

The Poynting Vector and the Ray Velocity Although the propagation vector \mathbf{k} defines the direction of the planes of constant phase for light waves in a crystal, the actual direction of the energy flow $\mathbf{E} \times \mathbf{H}$ is not in the same direction, generally, as that of \mathbf{k} . This stems from the fact that in anisotropic media \mathbf{E} and \mathbf{k} are not, in general, mutually perpendicular, as can be seen by inspection of Equation (6.91). On the other hand the magnetic field \mathbf{H} is perpendicular to both \mathbf{E} and \mathbf{k} because of the relation $\mathbf{k} \times \mathbf{E} = \mu_0 \omega \mathbf{H}$, which comes from the first Maxwell equation. The situation is shown graphically in Figure 6.11. The three vectors \mathbf{E} , \mathbf{k} , and $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ are all perpendicular to \mathbf{H} and, further, \mathbf{E} is perpendicular to \mathbf{S} .

Consider a narrow beam or ray of light in a crystal. The planes of constant phase are, perpendicular to \mathbf{k} , but they move along the direction of the ray \mathbf{S} . Thus the planes of constant phase are inclined to their direction of motion as shown in the figure. Let θ denote the angle between \mathbf{k} and \mathbf{S} . Then the surfaces of constant phase move with a velocity u —called the *ray velocity*—along the ray direction.

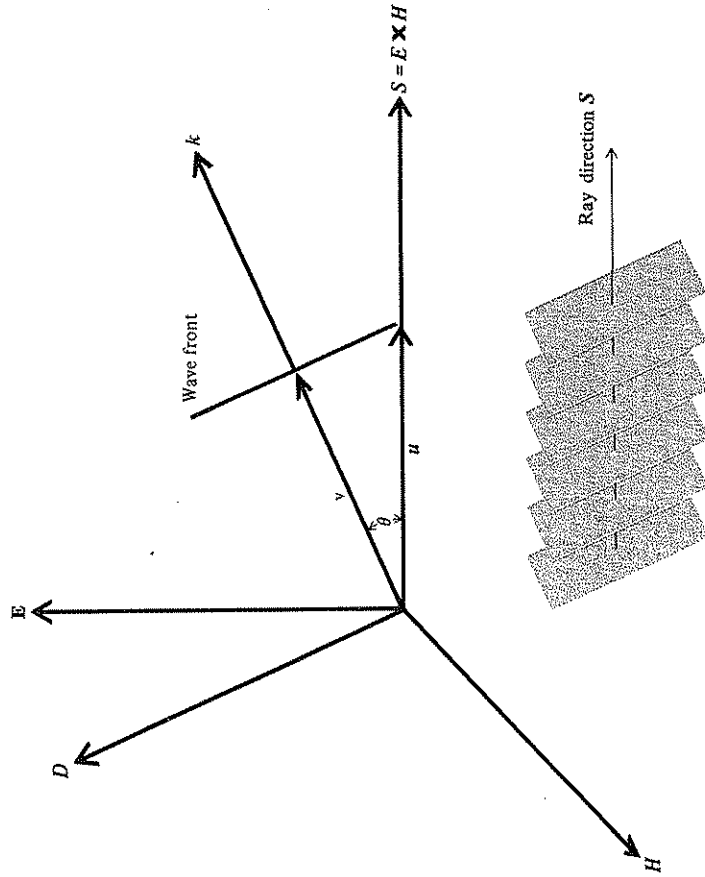


Figure 6.11. Relationships among the electric and magnetic fields, the Poynting vector, the wave vector, the ray-velocity vector, and the phase-velocity vector for plane waves in a crystal.

The magnitude of u is given by

$$u = \frac{v}{\cos \theta} \tag{6.106}$$

where v is the phase velocity (in the direction of k). Evidently the ray velocity is greater than the phase velocity except, of course, when $\theta = 0$. In the latter case the phase and ray velocities are equal. This occurs when the direction of propagation is along one of the principal axes of the crystal. In this case S and k also have the same direction.

The Ray-Velocity Surface This surface gives the magnitude of the ray velocity for any given direction of the ray. To find the equation of the ray-velocity surface, it will be convenient to express the wave Equation (6.91) in terms of the displacement vector $D = \epsilon_0(1 + \chi)E$. We have, then, for plane harmonic waves

$$k \times (k \times E) = -\frac{\omega^2}{c^2 \epsilon_0} D$$

This shows that D is perpendicular to the wave vector k . Expanding the triple product gives

$$k(k \cdot E) - k^2 E = -\frac{\omega^2}{c^2 \epsilon_0} D$$

Next, take the dot product with D . Then, since $k \cdot D = 0$, we find

$$k^2 E \cdot D = \frac{\omega^2}{c^2 \epsilon_0} D \cdot D$$

or, in view of the fact that $v = \omega/k$,

$$E \cdot D = ED \cos \theta = \frac{v^2}{c^2 \epsilon_0} D^2$$

Now if the coordinate axes are principal axes of the crystal, the components of E are related to those of D by

$$\epsilon_0 E_x = \frac{D_x}{\epsilon_{11}}, \frac{D_x}{n_1^2}$$

and similarly for the y and z components. Consequently, the equation for $E \cdot D$ above is equivalent to the following three scalar equations:

$$\begin{aligned} D_x \left(\frac{c^2}{n_1^2} - u_y^2 - u_z^2 \right) + D_y u_x u_y + D_z u_x u_z &= 0 \\ D_x u_y u_x + D_y \left(\frac{c^2}{n_2^2} - u_x^2 - u_z^2 \right) + D_z u_y u_z &= 0 \\ D_x u_x u_x + D_y u_y u_y + D_z \left(\frac{c^2}{n_3^2} - u_x^2 - u_y^2 \right) &= 0 \end{aligned}$$

The determinant of the coefficients must vanish in order that a non-trivial solution exists. This gives the equation of the ray-velocity surface,

$$\begin{vmatrix} c^2/n_1^2 - u_y^2 - u_z^2 & u_x u_y & u_x u_z \\ u_y u_x & c^2/n_2^2 - u_x^2 - u_z^2 & u_y u_z \\ u_x u_x & u_x u_y & c^2/n_3^2 - u_x^2 - u_y^2 \end{vmatrix} = 0 \tag{6.107}$$

In particular, the equations of the intercepts in the xy plane are obtained by setting $u_z = 0$. The result gives a circle

$$u_x^2 + u_y^2 = \frac{c^2}{n_3^2} \tag{6.108}$$

and an ellipse

$$n_2^2 u_x^2 + n_1^2 u_y^2 = c^2 \tag{6.109}$$

Corresponding equations can be obtained for the other coordinate planes by cyclic permutation, and in each case the intercepts consist of an ellipse and a circle. It is easily verified that the intercepts of the

treated refraction at a dielectric boundary, the law of refraction was seen to be contained in the equation

$$\mathbf{k}_0 \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{r} \quad (\text{at boundary}) \quad (6.110)$$

This same relation is also true for refraction at a boundary of a crystal since it merely expresses the fact that some unspecified boundary condition can exist at all. The equation implies that the projections of the propagation vectors along the boundary plane must be equal for both the incident and refracted waves. Now we know that for a given direction of propagation in the crystal, there are two possible propagation vectors. Owing to the double nature of the k surface, it is also true that for a prescribed value of the *projection* of the propagation vector in any given direction there are again two possible propagation vectors. This results in double refraction of a wave incident on the surface of a crystal as shown in Figure 6.13. From Equation (6.110),

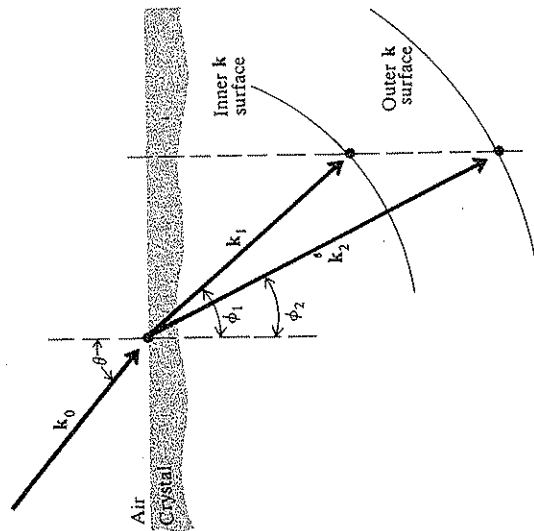


Figure 6.13. Wave vectors for double refraction at the boundary of a crystal.

we can write

$$k_0 \sin \theta = k_1 \sin \phi_1 \quad k_0 \sin \theta = k_2 \sin \phi_2 \quad (6.111)$$

for the two refracted waves.

At first sight it may appear that the above equations constitute a statement of Snell's law for double refraction. However, this is not the case. The trouble is that k_1 and k_2 are not constant in general,

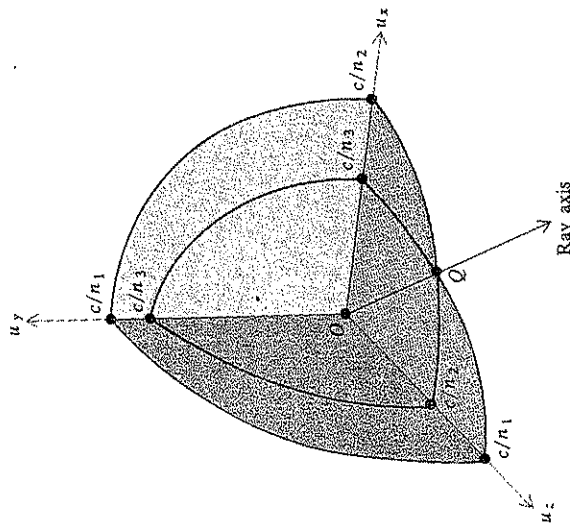


Figure 6.12. The ray-velocity surface.

ray-velocity surface along the coordinate axes are the same as those of the phase-velocity surface. Figure 6.12 shows the form of the ray-velocity surface. As with the phase-velocity surface, the ray-velocity surface consists of two sheets, an inner one and an outer one, corresponding to the two possible values of u for a given ray direction. The two sheets touch at a point Q that defines a direction for which the two ray velocities are equal. This direction is called the *ray axis* of the crystal.

In biaxial crystals there are two ray axes, and these ray axes are distinct from the optic axes of the crystal. On the other hand, in the case of a uniaxial crystal the two sheets of the ray-velocity surface consist of a sphere and an ellipsoid of revolution (spheroid). The two surfaces are tangent at the ends of a particular diameter of the sphere. This diameter defines the ray axis that, for uniaxial crystals, also coincides with the optic axis of the crystal.

6.8 Double Refraction at a Boundary

Consider a plane wave incident on the surface of a crystal. Denote the propagation vector of the incident wave by \mathbf{k}_0 and that of the refracted wave by \mathbf{k} , and call θ and ϕ the angles of incidence and refraction. According to the arguments in Section 2.6, where we