Maxwell Equations in the vacuum (Differential form)

$\vec{\nabla} \cdot \vec{E} = 0$ (field arises from an external source)

$\vec{\nabla} \cdot \vec{B} = 0$ (always, so $\vec{B} = \vec{\nabla} \times \vec{A}$)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday; so } \vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t\text{)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{(Ampere; field arises from an external source)}$$

$\epsilon_0 = 8.8542 \times 10^{-12}\, C^2/Nm^2$ permittivity of free space

$\mu_0 = 4\pi \times 10^{-7}\, Ns^2/C^2$ permeability of free space

$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \equiv 2.99792458 \times 10^8\, m/s$

Wave equations

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Use $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \xrightarrow{\vec{\nabla} \cdot \vec{E} = 0} -\nabla^2 \vec{E}$

$$\text{so } -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t}$$

Also, $\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$ and $\nabla^2 \vec{A} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$

<u>Solutions</u>: Plane waves $\vec{E}(\vec{r},t) = \vec{\mathcal{E}}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, $k = \omega/c$

$\vec{B}(\vec{r},t) = \vec{\mathcal{B}}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

Also, cylindrical waves, spherical waves and linear combinations.

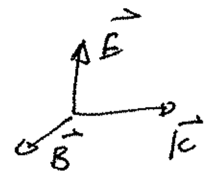
②

Relations among $\vec{E}$, $\vec{B}$ and $\vec{k}$.

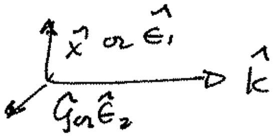
$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{E} = \omega\vec{B}, \text{ so } |\vec{B}| = \frac{k}{\omega}|\vec{E}| = \frac{|\vec{E}|}{c}$$

and



Polarization:



Linear basis

<u>orthonormal</u>

$\hat{\epsilon}_1, \hat{\epsilon}_2$ are unit vects.

$|\hat{\epsilon}_1|^2 = |\hat{\epsilon}_2|^2 = 1$

$\hat{\epsilon}_1 \cdot \hat{\epsilon}_2 = 0$

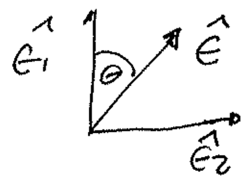
$\vec{E} = \mathcal{E}\hat{\epsilon}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ (use $\vec{k}\cdot\vec{r} = kz$ here)

<u>Linear polarized:</u>

$\hat{\epsilon} = a_1\hat{\epsilon}_1 + a_2\hat{\epsilon}_2$ where $|\hat{\epsilon}|^2 = 1 \Rightarrow \hat{\epsilon}\cdot\hat{\epsilon}^* = a_1^2|\hat{\epsilon}_1|^2 + a_2^2|\hat{\epsilon}_2|^2$

So $a_1^2 + a_2^2 = 1$

use angle θ



$a_1 = \cos\theta$

$a_2 = \sin\theta$

<u>Unpolarized light</u>

Equal fields ~~at~~ along $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ directions,

but <u>no phase relationship</u> ~~among~~ between these two fields.

<u>Circularly polarized light</u>

consider the circular basis $\{\hat{\epsilon}_+, \hat{\epsilon}_-\}$

$\hat{\epsilon}_+ = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 + i\hat{\epsilon}_2)$ $\quad \hat{\epsilon}_- = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 - i\hat{\epsilon}_2)$

"+" or <u>positive helicity</u> $\Rightarrow \vec{E} = \mathcal{E}\hat{\epsilon}_+ e^{i(kz-\omega t)}$

So $\vec{E} = \frac{\mathcal{E}}{\sqrt{2}}(\hat{\epsilon}_1 + i\hat{\epsilon}_2)e^{i(kz-\omega t)} = \frac{\mathcal{E}}{\sqrt{2}}\hat{\epsilon}_1 e^{i(kz-\omega t)} + \frac{\mathcal{E}}{\sqrt{2}}\hat{\epsilon}_2 e^{i(kz-\omega t)} e^{i\pi/2}$

Note $\hat{\epsilon}_2$ component is advanced in phase by $\pi/2$

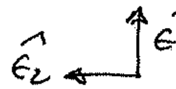
~~So if instantaneously the $\hat{\epsilon}_1$ component is "1" (maximum)~~

(4)

Consider the real field components

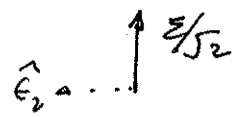
$$\vec{E} = \frac{\mathcal{E}}{\sqrt{2}}\,\hat{\epsilon}_1 \cos(kz-\omega t) + \frac{\mathcal{E}}{\sqrt{2}}\,\hat{\epsilon}_2 \cos(kz-\omega t+\pi/2)$$

Pictures at $\underline{t=0}$



looking along $-\hat{k}$

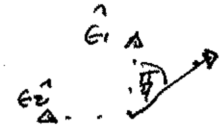
$kz=0$



$\hat{\epsilon}_1$ only

(Since $\cos\pi/2=0$)
$\hat{\epsilon}_2=0$

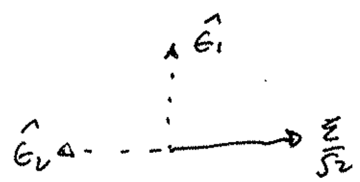
$kz=\pi/4$



$\hat{\epsilon}_1$ component is $\frac{\mathcal{E}}{\sqrt{2}}\frac{1}{\sqrt{2}}$ since $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

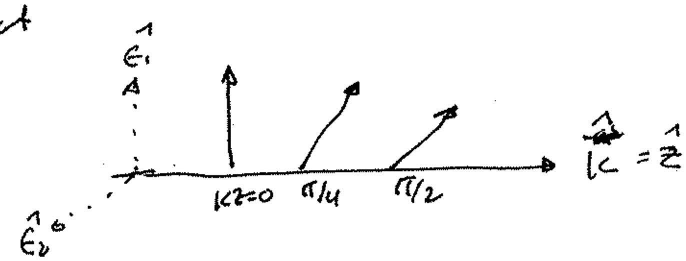
$\hat{\epsilon}_2$ compont is $-\frac{\mathcal{E}}{\sqrt{2}}\frac{1}{\sqrt{2}}$ since $\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}$

$kz=\pi/2$



since $\hat{\epsilon}_1$ component is ϕ
$\hat{\epsilon}_2$ " is -1

Snapshot



Now turn on time: the phase point $kz-\omega t=0$ moves at speed c past observer along $\hat{k}$ axis

So the Electric field appears to rotate to the left

Left-handed circular polarization.

Conclusion:

$\hat{\epsilon}_+$ is Left-handed, $\hat{\epsilon}_-$ is righthanded

Either basis can be used to describe any state of polarization

Example: Describe $\hat{\epsilon}_1$ linearly polarized light in the circular basis

$$\hat{\epsilon}_1 = \frac{\hat{\epsilon}_+ + \hat{\epsilon}_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{2\hat{\epsilon}_1}{\sqrt{2}} = \hat{\epsilon}_1$$

and
$$\hat{\epsilon}_2 = \frac{\hat{\epsilon}_+ - \hat{\epsilon}_2}{i\sqrt{2}} = \frac{1}{i\sqrt{2}}\frac{1}{\sqrt{2}}\, 2i\,\hat{\epsilon}_2 = \hat{\epsilon}_2$$

Orthogonality relations

$$\hat{\epsilon}_1 \cdot \hat{\epsilon}_2 = 0 \qquad \hat{\epsilon}_1 \cdot \hat{\epsilon}_1 = 1 = \hat{\epsilon}_2 \cdot \hat{\epsilon}_2$$

$$\hat{\epsilon}_+ \cdot \hat{\epsilon}_+^* = 1 = \hat{\epsilon}_- \cdot \hat{\epsilon}_-^* \equiv |\hat{\epsilon}_\pm|^2$$

$$\hat{\epsilon}_+ \cdot \hat{\epsilon}_+ = 0 = \hat{\epsilon}_- \cdot \hat{\epsilon}_-$$

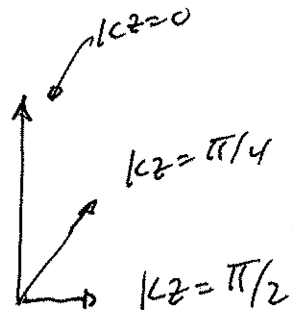
$$\hat{\epsilon}_+ \cdot \hat{\epsilon}_-^* = 0 = \hat{\epsilon}_+^* \cdot \hat{\epsilon}_-$$

$$\hat{\epsilon}_+ \cdot \hat{\epsilon}_- = 1 = \hat{\epsilon}_+^* \cdot \hat{\epsilon}_-^*$$

(6)

Elliptical polarization

Pictures at $t=0$



Pos. helicity but unequal amplitudes

$$\vec{E} = \mathcal{E}_1 \hat{\epsilon}_1 e^{i(kz-\omega t)} + \mathcal{E}_2 \hat{\epsilon}_2 e^{i(kz-\omega t+\pi/2)}$$

Look at Real part

$$\vec{E} = \mathcal{E}_1 \hat{\epsilon}_1 \cos(kz-\omega t) + \mathcal{E}_2 \hat{\epsilon}_2 \cos(kz-\omega t+\pi/2)$$

~~Let~~

Pos. helicity: $\vec{E} = \frac{\mathcal{E}}{\sqrt{1+a}}\left(\hat{\epsilon}_1 + ia\,\hat{\epsilon}_2\right) e^{i(kz-\omega t)}$

where $0 < a \leq 1$

Neg. helicity: $\vec{E} = \frac{\mathcal{E}}{\sqrt{1+a}}\left(\hat{\epsilon}_1 - ia\,\hat{\epsilon}_2\right) e^{i(kz-\omega t)}$

General Elliptical

Radiation from an oscillating dipole moment

A dipole moment for the charge distribution $\begin{matrix} +q \\ -q \end{matrix} \uparrow \vec{r}$

is $\vec{P}_0 = q\vec{r}$.

An oscillating dipole moment is described by $\vec{p}(t) = \vec{P}_0 \cos\omega t$ or $\vec{P}(t) = \vec{P}_0\, e^{-i\omega t}$. The latter will be used.

Without presenting a derivation, the radiated vector potential from a single oscillating point dipole is

$$\vec{A}(\vec{r},t) = -ik\,\vec{p}(t)\,\frac{e^{ikr}}{r}\,\frac{\mu_0 c}{4\pi}$$

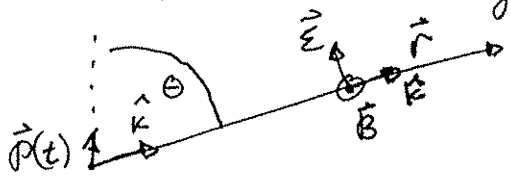
The magnetic field is $\vec{B}(\vec{r},t) = k^2\frac{\mu_0 c}{4\pi}(\hat{k}\times\vec{p}(t))\,\frac{e^{ikr}}{r} = \vec{\nabla}\times\vec{A}$

And the electric field is $\vec{E}(\vec{r},t) = k^2(\hat{k}\times\vec{p}(t))\times\hat{k}\,\frac{e^{ikr}}{r}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{c}{4\pi}$

$$= c\vec{B}\times\hat{k} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{\vec{B}}{\mu_0}\times\hat{k}$$

A point dipole is one which is very small compared the wavelength of the emitted radiation and which is very small compared to the distance from the dipole to the position of observation

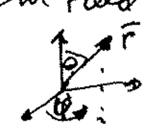
The unit vector $\hat{k}$ is in the direction of observation $\hat{r}$. Remember that the fields are always $\perp$ to $\hat{k}$.



$\hat{k}\times\vec{p}(t) = \hat{k}\times\vec{p}\,e^{-i\omega t}$

$\hat{k}\times\vec{p} = kp\sin\theta\;\hat{b}$ (RHR)

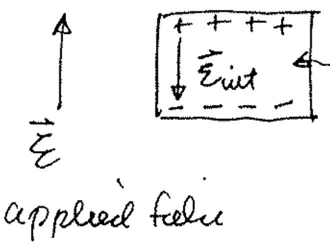
Thus, the $\frac{e^{ikr}}{r}$ factor is easily understood as an outgoing spherical wave, while the $\hat{k}\times\vec{p}$ factor introduce a Θ dependence to the strength of the EM field. Note that the field is independent of the angle φ.



(8) Response of matter to an applied EM field

Matter is "polarizable", which means that a very small volume will exhibit a dipole moment $\vec{p}$ in response to an applied static electric field $\vec{\mathcal{E}}$.

A conductor is "infinitely polarizable" in the sense that ~~of~~ an applied field will push the charges within the metal to the surfaces.



In a metal, $\vec{\mathcal{E}}_{internal} = -\vec{\mathcal{E}}$

so there is no net field within the metal

$\vec{\mathcal{E}}_{net} = \vec{\mathcal{E}} + \vec{\mathcal{E}}_{int} = 0$

In a "dielectric" material, the charges are not free to move very much, so the induced dipole moments are much smaller. A simple and effective way to model the response of a dielectric material to an EM field is to consider the "polarizable electrons" in the material to be electrons attached to massive nuclei via a spring

So, each atom or molecule becomes a simple  system

The spring constant will be k and the "effective mass" will be m and the "effective charge" will be $-e$.

A single oscillator at the position $\vec{r} = 0$ is subjected to the field $\vec{E}(\vec{r}=0, t) = \vec{\mathcal{E}} e^{-i\omega t}$. The displacement of the charge from the equilibrium position will be $\vec{d}(t) = \vec{d}_0 e^{-i\omega t}$. When $\vec{d} = 0$, the induced dipole moment is 0. $\vec{d}_0$ may be complex to allow for a phase shi[ft]

The equation of motion is $m\ddot{\vec{d}} = -e\vec{E} - k\vec{d} - \gamma m\dot{\vec{d}}$. γ is represe[ntative] of friction or damping.

$$\dot{\vec{d}} = -i\omega\vec{d} \quad , \quad \ddot{\vec{d}} = -\omega^2\vec{d}$$

so $\quad -\omega^2 m\vec{d} = -e\vec{\mathcal{E}} - k\vec{d} + i\omega\gamma m\vec{d}$

or $(k-\omega^2 m - i\omega\gamma_m)\vec{d} = -e\vec{E} \Rightarrow \vec{d} = \dfrac{-e\vec{E}}{(k - m\omega^2 - i\gamma_m\omega)}$

So $\vec{d_0} = \dfrac{-e\vec{\mathcal{E}}}{(k - m\omega^2 - i\gamma_m\omega)}$

Now need the dipole moment $\vec{p}(t) = -e\vec{d} = \dfrac{e^2\vec{E}}{m\left(\frac{k}{m} - \omega^2 - i\omega\gamma\right)}$

where $k/m = \omega_0^2$ = square of the natural frequency.

Hence $\vec{p}(t) = \dfrac{e^2\vec{E}(t)}{m\left((\omega_0^2-\omega^2) - i\gamma\omega\right)} \Rightarrow \vec{p_0} = \dfrac{e^2\vec{\mathcal{E}}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$

Using N as the density of oscillators, the polarization becomes

$\vec{P} = N\vec{p} = \dfrac{Ne^2\vec{E}(t)}{m(\omega_0^2-\omega^2-i\gamma\omega)}$ if all the electrons behave the same way.

Then $\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \Rightarrow \epsilon = \epsilon_0 + \dfrac{Ne^2}{m(\omega_0^2-\omega^2-i\gamma\omega)}$

Note that the electric susceptibility is then

$$\chi = \frac{Ne^2}{m(\omega_0^2-\omega^2-i\gamma\omega)}$$

We need the dielectric function K_e from $\epsilon = \epsilon_0 K_e$.

$$K_e(\omega) = \frac{Ne^2}{\epsilon_0 m(\omega_0^2-\omega^2-i\gamma\omega)} + 1$$

Generalizing to a material which may have electrons of different effective charges, ~~or~~ of different effective masses and of different spring constants we can write

$$K_e(\omega) = \sum_j \frac{N_j e_j^2}{\epsilon_0 m_j(\omega_{0j}^2 - \omega^2 - i\gamma_j\omega)} + 1$$

(10) The index of refraction is defined by $v^2 = \frac{1}{\epsilon\mu} = \frac{1}{\epsilon_0\mu_0 K_e K_m} = \frac{c^2}{K_e K_m} \equiv \frac{c}{V}$.

So $n = \sqrt{K_e K_m}$. We will assume that $K_m = 1$.

This v is the phase velocity $v = \omega/k$.

Note that $\vec{K}\cdot\vec{z} = \frac{\omega}{v} z = n\frac{\omega}{c} z = n K_{VAC} z$ if $\vec{K}$ is in the $\vec{z}$ direction

Some particular cases of n.

1) Free electron plasma. $\omega_0 = 0$, $\gamma = 0$ for all relevant electrons

$$K_e = 1 + \frac{Ne^2}{\epsilon_0 m(-\omega^2)} = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{where } \omega_p = \sqrt{\frac{Ne^2}{\epsilon_0 m}} \text{ is the}$$

"plasma frequency"

So n = real number for $\omega > \omega_p$ but n = imaginary for $\omega < \omega_p$

2) Metal with $\gamma \simeq 10^{17}$ Hz with ω for green light
λ_{green} (in vacuum) $\simeq 500$ nm $\Rightarrow \nu = \frac{c}{\lambda} = \frac{3\times10^{8}\text{ m/s}}{5\times10^{-7}\text{ m}} = 6\times10^{14}$ Hz

so $\omega_{green} = 2\pi\nu \simeq 3.6\times10^{15}$ Hz. And $\omega_0 \simeq 0$

then $K_e = n^2 \simeq 1 + \frac{Ne^2}{\epsilon m(-i\gamma\omega)}$ since $\gamma\omega > \omega^2$.

or $n^2 = 1 + \frac{iNe^2}{\epsilon_0 m\gamma\omega} \equiv 1 + i\frac{\sigma}{\omega}$, $\sigma = \frac{Ne^2}{\epsilon_0 m\gamma} \equiv$ conductivity

3) Metal at a high frequency such that $\omega_0 \simeq 0$, $\omega >> \gamma$.

$$n^2 = K_e = 1 - \frac{\omega_p^2}{\omega^2}$$

Maxwell's Eqs in matter

Assume that there are no free charges or currents in the material

$\vec{\nabla}\cdot\vec{D} = 0$ where $\vec{D} = \epsilon\vec{E}$ is a constitutive relation which embodies the response of the medium to the oscillating electric field

or $\vec{\nabla}\cdot\vec{E} = 0$

$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$, where $\vec{P}$ is the "polarization" or "dipole moment density". An electric field can instantaneously induce dipole moments throughout the medium. A dipole moment is a separation of charge $\underset{q}{\ominus}\xrightarrow{\vec{r}}\underset{q}{\oplus}$ and is defined to be the quantity $q\vec{r}$. The density of such dipole moments is $\vec{P}$. $\vec{P} = \chi\vec{E}$, where χ is the electric susceptibility of the material. Thus $\vec{D} = \epsilon_0\vec{E} + \vec{P} = (\epsilon_0 + \chi)\vec{E} = \epsilon\vec{E}$, and $\epsilon = \epsilon_0 + \chi = \epsilon_0(1 + \chi/\epsilon_0) = \epsilon_0 K_e$. K_e is a number > 1. It is actually a function of the frequency of the EM field. Thus $\epsilon(\omega) = \epsilon_0 K_e(\omega)$. $K_e(\omega)$ is called the dielectric function of the material

$\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow \vec{D}\perp\vec{k}$, using $\vec{D} = \vec{\mathcal{D}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ with $\vec{\mathcal{D}} = \epsilon\vec{\mathcal{E}}$.

$\vec{\nabla}\cdot\vec{B} = 0$ So $\vec{B} = \vec{\mathcal{B}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is $\perp\vec{k}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ $\Rightarrow$ $\vec{B} = \hat{k}\times\vec{E}/c$, so $\vec{B}$ is $\perp\vec{E}$

(12) $\vec{\nabla} \times \vec{B} = \mu \epsilon \frac{\partial}{\partial t} \vec{E}$

or $\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial}{\partial t} \vec{E}$

Here $\mu = \mu_0 + \chi_m$ is the permeability of the material, and $\mu = \mu_0 K_m$, $K_m = 1 + \chi_m/\mu_0$. K_m is also a function of ω. The underlying constitutive relation is $\vec{B} = \mu \vec{H}$. $\vec{H}$ is the "auxiliary field" which is useful for mathematical reasons. Also, $\vec{B} = \mu_0 \vec{H} + \vec{M}$, where $\vec{M}$ is the "magnetization" or magnetic dipole density. $\vec{M} = \chi_m \vec{H}$ with χ_m being the magnetic susceptibility. Hence, $\vec{B} = \mu_0 \vec{H} + \chi_m \vec{H} \Rightarrow \mu_0 (1 + \chi_m/\mu_0) = \mu_0 K_m$.

All four ME's yield the wave equations describing EM waves propagating in the medium:

$$\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2}{\partial t^2} \vec{E} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \vec{E}$$

and $$\nabla^2 \vec{B} = \mu \epsilon \frac{\partial^2}{\partial t^2} \vec{B} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \vec{B}$$

where $$v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{1}{\sqrt{\mu_0 \epsilon_0} \sqrt{K_e K_m}} = \frac{c}{\sqrt{K_e K_m}} \equiv \frac{c}{n}$$

with n = index of refraction, a function of ω.

Using plane waves as solutions, we find that $\nabla^2 \vec{E} = -k^2 \vec{E}$ and $\frac{\partial^2}{\partial t^2} \vec{E} = -\omega^2 \vec{E}$. So, $\nabla^2 \vec{E} = \frac{k^2}{\omega^2} \frac{\partial^2}{\partial t^2} \vec{E}$.

Hence $$\frac{k^2}{\omega^2} = \frac{1}{v^2} = \frac{n^2}{c^2} \Rightarrow k = \frac{n\omega}{c}.$$

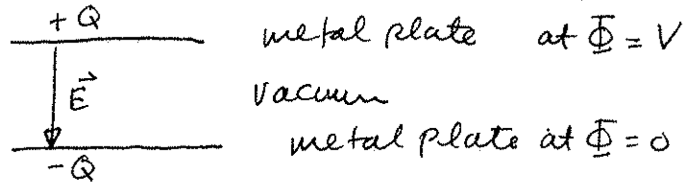
But since $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\nu$ and $\lambda_{VAC}\nu = c$ in vacuum,

$$\frac{1}{\lambda} = \frac{n\nu}{c} = \frac{n}{\lambda_{VAC}}, \text{ or } \lambda \text{ (in medium)} = \frac{\lambda \text{ (in vacuum)}}{n}$$

Energy Density of an electromagnetic field

Consider a capacitor

Ignore edge effects



Φ is the electrostatic potential measured in Volts. $\vec{E}$ is measured in V/m.

If you move a charge q through a change in potential $\Delta\Phi$, the work you have to perform is $q\Delta\Phi$. Thus, $dW = q\,d\Phi$. But the charge stored in the capacitor is $q = C\Phi$, where C is the capacitance and Φ is the potential across the plates. So,
$dW = C\Phi\,d\Phi \Rightarrow W = \int_0^V C\Phi\,d\Phi = \frac{1}{2}CV^2$. The capacitance

is $C = \epsilon_0 A/d$, so $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 = \frac{1}{2}\epsilon_0 Ad\left(\frac{V}{d}\right)^2 = \frac{1}{2}\epsilon_0 E^2 \cdot \text{volume}$

Since W must be the energy stored in the capacitor, the energy density must be $W/\text{volume} = \frac{1}{2}\epsilon_0 E^2 \equiv u_e$

If the material between the plates has a dielectric function ϵ, then $u_e = \frac{1}{2}\epsilon E^2$ is the energy density of the electric field.

Similarly, an analysis involving the generation of currents leads to the energy density of the magnetic field which is
$u_m = \frac{1}{2}\frac{B^2}{\mu_0}$ in vacuum or $\frac{1}{2}\frac{B^2}{\mu}$ in matter.

We can use these expressions to determine the energy density of a propagating EM field:

$$u_e = \frac{1}{2}\epsilon E^2 \text{ and } u_m = \frac{1}{2}\frac{B^2}{\mu} = \frac{1}{2\mu}\frac{E^2}{v^2} = \frac{1}{2}\frac{\mu\epsilon}{\mu}E^2 = u_e.$$

The total energy density is $u_e + u_m = \epsilon E^2 = B^2/\mu = u$.

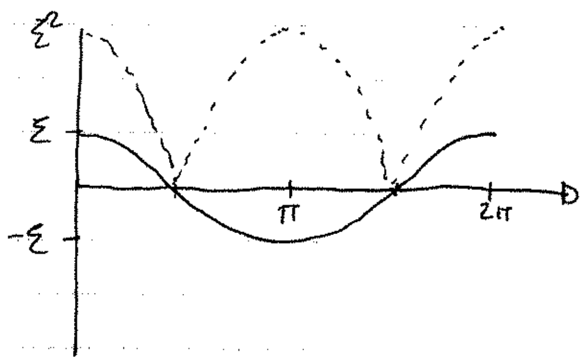
For complex, time-harmonic fields $u = \epsilon\frac{\vec{E}\cdot\vec{E}^*}{2} = \frac{\vec{B}\cdot\vec{B}^*}{2}/\mu$ (time-averaged)

14 Power per unit area carried by an EM plane wave

Begin with the energy density $u = \epsilon |\vec{E}|^2 = |\vec{B}|^2/\mu$

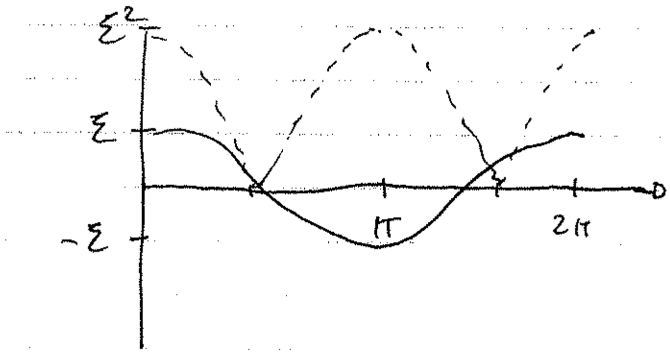
For the moment choose $t=0$, so $\vec{E} = \mathrm{Re}\, \vec{\mathcal{E}} e^{i\vec{k}\cdot\vec{r}}$, $\vec{B} = \mathrm{Re}\, \vec{\mathcal{B}} e^{i\vec{k}\cdot\vec{r}}$

with $\vec{k} = k\hat{z}$. Plot this real $\vec{E}$ as a function of $\vec{k}\cdot\vec{r} = kz$,



$\vec{k}\cdot\vec{r} = kz$ As kz varies from 0 to 2π z varies from 0 to λ

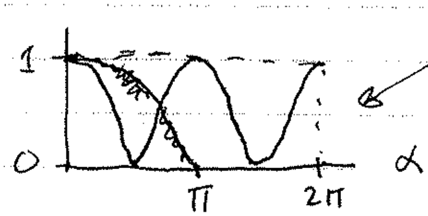
Or we can choose $\vec{k}\cdot\vec{r} = 0$, so $\vec{E} = \mathrm{Re}\, \vec{\mathcal{E}} e^{-i\omega t} = \vec{\mathcal{E}} \cos\omega t$



ωt

Now consider a box of volume $A\lambda$, where A is an area in the xy plane, moving by. The total energy passing the point $z=0$ per unit ~~energy~~ area during one period of the wave (period $= \frac{1}{\nu} = 2\pi/\omega$). v is the speed.

$$U = \int_0^{2\pi/\omega} v\, u(t)\, dt = \epsilon \mathcal{E}^2 v \int_0^{2\pi/\omega} \cos^2 \omega t\, dt = \epsilon \mathcal{E}^2 v \int_0^{2\pi} \cos^2 \alpha \frac{d\alpha}{\omega}$$



area of rectangle $= 2\pi = \int_0^{2\pi} (\cos^2\alpha + \sin^2\alpha)\, d\alpha$

So $\int_0^{2\pi} \cos^2\alpha\, d\alpha = \int_0^{2\pi} \sin^2\alpha\, d\alpha = \pi$

$U = \epsilon \mathcal{E}^2 \frac{\pi}{\omega} v$, So the energy/unit area/unit time $= \frac{U}{\text{period}} = \frac{U\omega}{2\pi} = \frac{\epsilon \mathcal{E}^2 v}{2}$

Time-averaged

This is called the magnitude of the Poynting Vector $S = \epsilon (\mathcal{E}^2/2) v$

Since energy flows in the direction of $\hat{k}$, $\vec{S} = S\hat{k}$.

But $\hat{\mathcal{E}} \times \hat{B} = \hat{k} \Rightarrow \vec{S} = \frac{\epsilon \mathcal{E}^2 v}{2} \hat{\mathcal{E}} \times \hat{B}$.

And $B = \mathcal{E}/v \Rightarrow \frac{\epsilon}{2} v^2 \mathcal{E}\hat{\mathcal{E}} \times B\vec{B} = \frac{\epsilon}{2\mu\epsilon} \vec{\mathcal{E}} \times \vec{B}$

Hence $\vec{S} = \frac{1}{2} \frac{\vec{\mathcal{E}} \times \vec{B}}{\mu}$ or $\cancel{\vec{S} = \frac{1}{2} \vec{E} \times \vec{B}^*}$

or $\vec{S} = \frac{1}{2} \frac{\vec{E} \times \vec{B}^*}{\mu}$ where both $\vec{E}$ and $\vec{B}$ are complex representations of the EM wave.

$= \frac{1}{2} \vec{E} \times \vec{H}^*$

$\boxed{\vec{S} \text{ is a time-averaged vector}}$

The irradiance is

$$I = S = \frac{\epsilon v \mathcal{E}^2}{2}. \quad I \text{ is in } W/m^2,\ \epsilon \text{ in } \frac{C^2}{Nm^2},\ \mathcal{E} \text{ in } \frac{V}{m}$$

Note that $\left[\frac{I}{\epsilon v}\right] = \frac{N^2}{C^2}$ but $N = \frac{CV}{m}$, so $\frac{N^2}{C^2} = \frac{V^2}{m^2}$

(C = Coulomb, V = volt, W = watt) $\left[\text{OR, } S = \frac{\epsilon}{2\sqrt{\epsilon\mu}} \mathcal{E}^2 = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 = \frac{1}{2}\frac{\mathcal{E}^2}{Z}\right]$

Momentum density: $\vec{m} = \vec{S}/v^2$, both are time-averaged quantities.

$$\text{units: } \frac{W/m^2}{m^2/s^2} = \frac{Nm/m^2}{\cancel{}\ m^2/s} = \frac{Ns}{m^3} = \frac{kg\, m/s}{m^3}$$

Radiation pressure $= \frac{d\vec{M}}{dt}/\text{area} = \vec{m} v = \frac{\vec{S}}{v}$

M = total momentum

If the collision between the EM wave and an object is elastic (totally reflecting) and at normal incidence, pressure $= 2S/v$

Example: $I = 1 \frac{W}{m^2} \Rightarrow$ pressure $= 3 \times 10^{-9} \frac{N}{m^2} = 3 \times 10^{-14}$ atm

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Maxwell's Equations of an EM field in a region of space free of sources of the field.

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \qquad \vec{\nabla}\times\vec{B} = \mu\epsilon\,\dot{\vec{E}}$$

where $\vec{D} = \underline{\epsilon}\,\vec{E}$ and $\dot{\vec{B}} = \frac{\partial}{\partial t}\vec{B}$, $\dot{\vec{E}} = \frac{\partial}{\partial t}\vec{E}$

Note that $\vec{\nabla}\cdot\vec{D} = \sum_i \epsilon_{ii}\frac{\partial E_i}{\partial r_i} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$

for $\underline{\epsilon}$ = constant, diagonal matrix

The wave equation:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

and $\vec{\nabla}\times(-\dot{\vec{B}}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial^2}{\partial t}\vec{E}\,\mu\epsilon = -\mu\epsilon\,\ddot{\vec{E}}$

So $\nabla^2\vec{E} = \mu\epsilon\,\ddot{\vec{E}} = \frac{1}{v^2}\ddot{\vec{E}}$

Now introduce a source for this time-varying EM field by adding a current density term to Ampere's Law

$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\,\dot{\vec{E}}$ This is need to describe a new field created when an external field drives a medium.

$\vec{J}$ arises from an oscillating polarization driven by some source

$\vec{D} = \epsilon\vec{E} + \vec{P}_S$, where $\vec{P}_S$ is the source polarization for the field $\vec{E}$.

And $\vec{\nabla}\times\vec{B} = \cancel{\mu\epsilon\dot{\vec{E}} =} \mu\dot{\vec{D}} = \mu(\epsilon\dot{\vec{E}} + \dot{\vec{P}}_S)$

or $\vec{\nabla}\times\vec{B} = \mu\dot{\vec{P}}_S + \mu\epsilon\,\dot{\vec{E}}$, so $\vec{J} = \dot{\vec{P}}_S$

Now, construct a new wave equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} \quad \text{, as before}$$

But $\vec{\nabla} \times (-\dot{\vec{B}}) = -\partial_t (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \{ \mu \dot{\vec{P}}_S + \mu \epsilon \dot{\vec{E}} \}$

So $\nabla^2 \vec{E} = \mu \dddot{\vec{P}}_S + \mu \epsilon \ddot{\vec{E}}$ (1)

We need to solve this equation given $P_S(\vec{r}, t)$ and some boundary condition on $\vec{E}$.

Assumptions:

1. $\vec{P}_S$ is uniform in space in amplitude and direction.

$$\vec{P}_S(\vec{r}, t) = \vec{P}_S \, e^{i(\varphi(\vec{r}) - \omega t)}$$

Then, using $\vec{E}(\vec{r}, t) = \vec{\mathcal{E}}(\vec{r}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$, the wave equation becomes

$$\nabla^2 \vec{E} = -\omega^2 \mu \vec{P}_S - \omega^2 \mu \epsilon \vec{E} \quad (2)$$

What is $\varphi(\vec{r})$? Usually, $\vec{P}_S(\vec{r}, t) = \chi^{(2)} \vec{E}_1(\vec{r}, t) \vec{E}_1(\vec{r} t)$

where $\vec{E}_1 = \vec{\mathcal{E}}_1 e^{i(\vec{k}_1 \cdot \vec{r} - \omega_1 t)}$

Then $\vec{P}_S = \chi^{(2)} \vec{\mathcal{E}}_1 \vec{\mathcal{E}}_1 e^{i(2\vec{k}_1 \cdot \vec{r} - \omega t)}$, so $\varphi(\vec{r}) = 2\vec{k}_1 \cdot \vec{r}$

Back to wave equation (2). Factor out $e^{-i\omega t}$

Then, $\nabla^2 \vec{\mathcal{E}}(\vec{r}) e^{i\vec{k} \cdot \vec{r}} = -\omega^2 \mu \vec{P}_S e^{i\varphi(\vec{r})} - \omega^2 \mu \epsilon \vec{\mathcal{E}}(\vec{r}) e^{i\vec{k} \cdot \vec{r}}$ (3)

Assumptions:

2. $\vec{k} \cdot \vec{r} = kz$
3. $\frac{\partial}{\partial x} \vec{\mathcal{E}}(\vec{r}) = 0$, $\frac{\partial}{\partial y} \vec{\mathcal{E}}(\vec{r}) = 0$

Thus, $\vec{E}$ is a growing or shrinking plane wave

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Eq (3) is now

$$\frac{\partial}{\partial z}\left\{\left(\frac{\partial}{\partial z}\mathcal{E}(z)\right)e^{ikz} + \mathcal{E}(z)\, ik\, e^{ikz}\right\} = -\omega^2\mu P_S e^{i\varphi} - \omega^2\mu\epsilon\, \mathcal{E}(z)\, e^{ikz}$$

or
$$\left(\frac{\partial^2}{\partial z^2}\mathcal{E}(z)\right)e^{ikz} + ik2\frac{\partial}{\partial z}\mathcal{E}(z)\, e^{ikz} - \mathcal{E}(z)\,k^2 e^{ikz}$$

$$= -\omega^2\mu P_S\, e^{i\varphi} - \omega^2\mu\epsilon\, \mathcal{E}(z)\, e^{ikz}$$

Assumption:

4. $\frac{\partial^2}{\partial z^2}\mathcal{E}(z) = 0$

Then multiply by e^{-ikz}

$$2ik\,\frac{\partial \mathcal{E}(z)}{\partial z} - k^2\,\mathcal{E}(z) = -\omega^2\mu P_S\, e^{i(\varphi - kz)} - \omega^2\mu\epsilon\,\mathcal{E}(z)$$

$$k^2 = \omega^2\mu\epsilon = \omega^2 n^2/c^2$$

so
$$2ik\,\frac{\partial \mathcal{E}(z)}{\partial z} = -\omega^2\mu P_S\, e^{i(\varphi - kz)}$$

Integrate:
$$\mathcal{E}(z) - \mathcal{E}(z_0) = \frac{i\omega^2\mu}{2k} P_S \int_{z_0}^{z} e^{i(\varphi - kz')}\, dz'$$

Assumption: 5. $\mathcal{E}(z_0) = 0$ and $z_0 = 0$

$$\mathcal{E}(z) = \frac{i\omega}{2}\sqrt{\frac{\mu}{\epsilon}}\, c\, P_S \int_{0}^{z} e^{i(\varphi - kz')}\, dz'$$

where $\sqrt{\frac{\mu}{\epsilon}}$ is the impedance

Assumption: 6. $\varphi = 2\vec{k}_1 \cdot \vec{r} = 2(k_{1x}x + k_{1y}y) + 2k_{1z}z$

Then $e^{i(\varphi - kz')} = e^{i2(k_{1x}x + k_{1y}y)} e^{i(2k_{1z}z' - kz')}$

$= f(x,y)\, e^{i(2k_{1z} - k)z'}$

Assumption: 7. $f(x,y) = 1$, $k_{1z} = k_1$, $k_{1x} = 0 = k_{1y}$

Then,
$$\mathcal{E}(z) = \frac{i\omega c}{2}\sqrt{\frac{\mu}{\epsilon}}\, P_S \int_0^z e^{i(2k_1 - k)z'} dz'$$

$$= \frac{i\omega c}{2}\sqrt{\frac{\mu}{\epsilon}}\, P_S \frac{1}{i\Delta k}\left\{ e^{i\Delta k z} - 1 \right\}, \text{ with } \Delta k = 2k_1 - k$$

Use $e^{i\Delta kz} - 1 = \left\{ e^{i\Delta kz/2} - e^{-i\Delta kz/2} \right\} e^{i\Delta kz/2}$

$= 2i \sin \Delta kz/2 \; e^{i\Delta kz/2}$

$$\mathcal{E}(z) = -\omega c \sqrt{\frac{\mu}{\epsilon}}\, P_S\, e^{i\Delta kz/2} \frac{\sin \Delta kz/2}{\Delta k}$$

$$= -\omega c \sqrt{\frac{\mu}{\epsilon}}\, P_S\, e^{i\Delta kz/2} \frac{z}{2} \frac{\sin \Delta kz/2}{\Delta kz/2}$$

Poynting Vector $\vec{S} = \hat{z} \frac{1}{2} |\mathcal{E}|^2 \frac{1}{\sqrt{\mu/\epsilon}}$

$$|S| = \frac{1}{2}\frac{1}{\sqrt{\mu/\epsilon}}\, \omega^2 c^2 \frac{\mu}{\epsilon}\, P_S^2 \frac{z^2}{4} \operatorname{sinc}^2 \Delta kz/2$$

$$= \frac{\omega^2 c^2}{8}\sqrt{\frac{\mu}{\epsilon}}\, P_S^2\, z^2 \operatorname{sinc}^2 \Delta kz/2$$

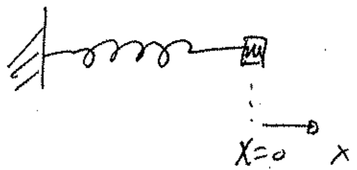
and $P_S^2 = |\chi^{(2)} \bar{\mathcal{E}}_1 \bar{\mathcal{E}}_1|$

$$\chi^{(2)} \vec{\mathcal{E}}_1 \vec{\mathcal{E}}_1 = \sum_i \hat{i}\, \chi_{ijk}\, \mathcal{E}_{1j}\, \mathcal{E}_{1k}$$

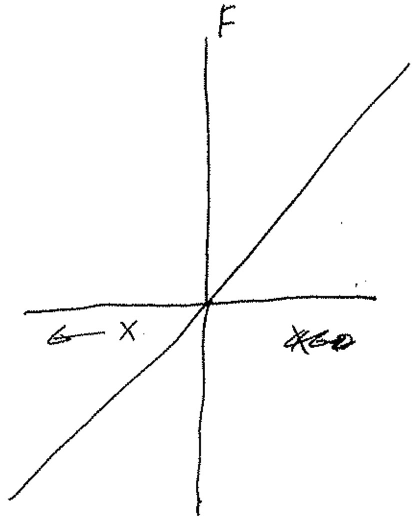
Harmonic oscillator

$F = -kx \Rightarrow F = -\frac{\partial U}{\partial x} \Rightarrow U(x) = \frac{1}{2}kx^2$

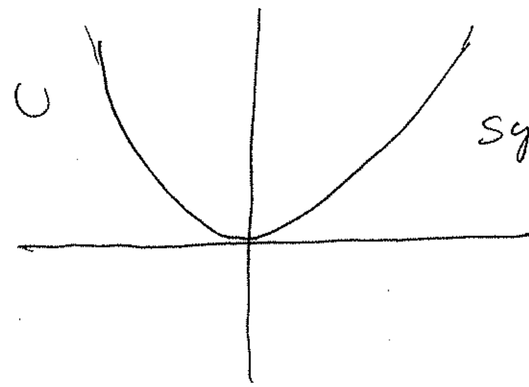
(20)



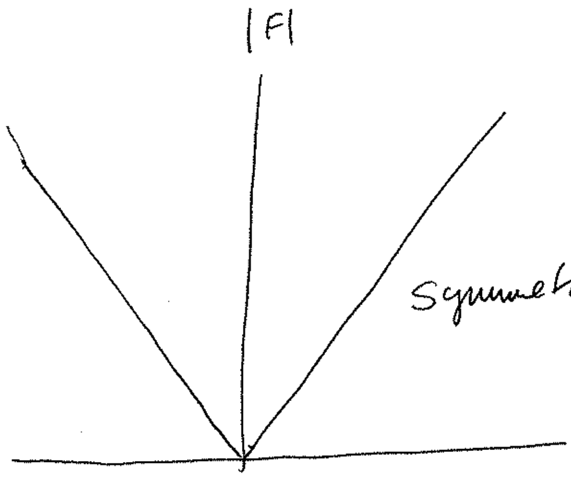
Harmonic for small vibration.



$\Rightarrow$



Symmetric



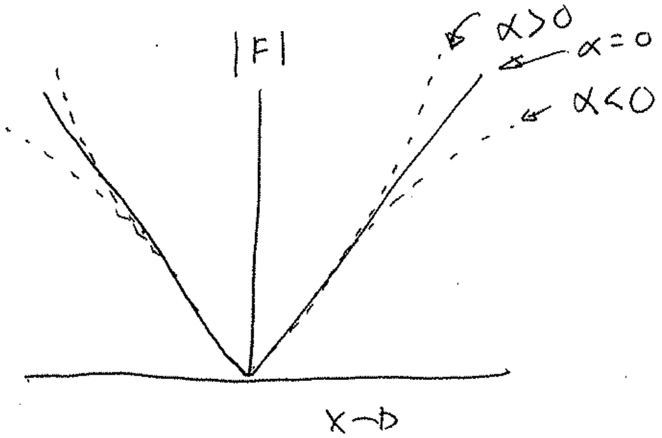
Eq. of motion (free oscillator) $\quad m\ddot{x} = -kx \Rightarrow x(t) = (a\sin\omega t + b\cos\omega t)$

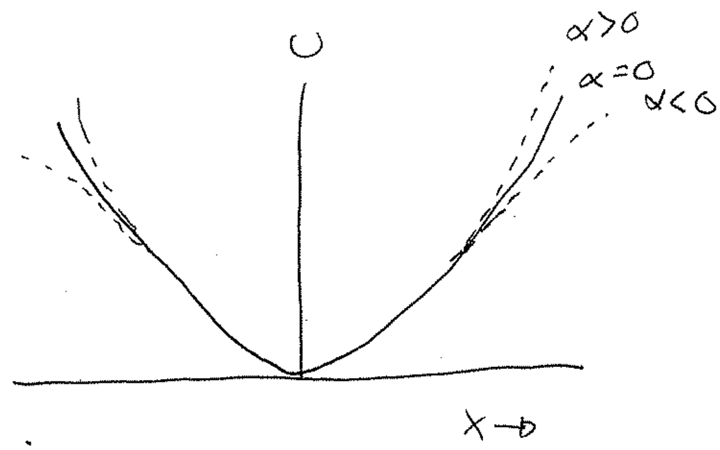
Driven oscillator $\quad m\ddot{x} = -kx + f(t) - \gamma\dot{x}$ damping term

plus boundary condition

Cubic Anharmonic oscillator

$$F = -Kx - \alpha x^3 \qquad F = -\frac{\partial U}{\partial x} \Rightarrow U(x) = \frac{1}{2}Kx^2 + \frac{\alpha}{4}x^4$$





$\alpha > 0 \leftrightarrow$ spring stiffens with extension

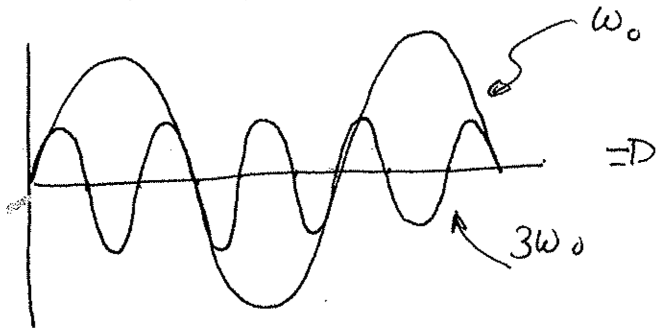
$\alpha < 0 \leftrightarrow$ spring softens with extension

Free oscillator:

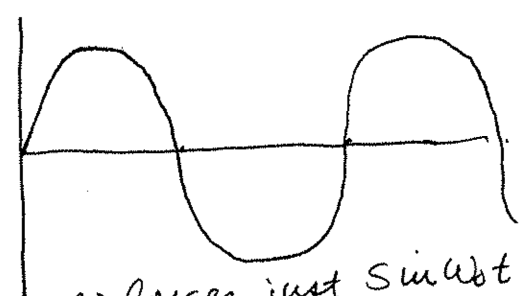
$$\left.\begin{array}{l} \alpha > 0 \Rightarrow \omega \gtrsim \omega_0 = \sqrt{\frac{k}{m}} \\ \alpha < 0 \Rightarrow \omega \lesssim \omega_0 = \sqrt{\frac{k}{m}} \end{array}\right\} \omega \doteq \left(1 + \frac{3}{8}\frac{\alpha}{k}x_0^2\right)\omega_0$$

① motion is periodic since energy is conserved

② motion is symmetric since potential is symmetric, force is symmetric.
$x(t+T/2) = -x(t)$ for T = period



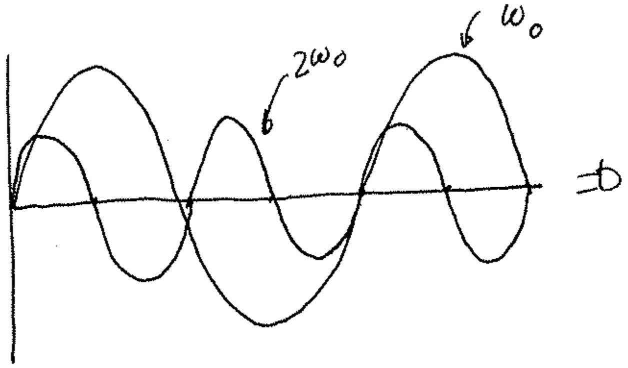
⇒



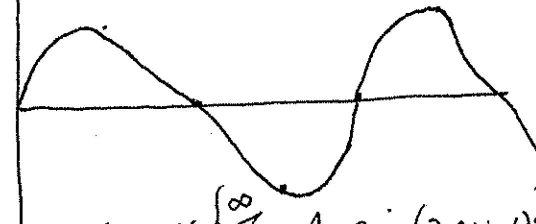
no longer just $\sin\omega_0 t$

$$x(t) = x_0 \sum_{n=0}^{\infty} A_n \sin((2n+1)\omega_0 t)$$

$$x(t+T/2) = -x(t)$$



⇒



$$x(t) = x_0\left\{\sum_{n=1}^{\infty} A_n \sin(2n\omega_0 t) + A_1 \sin\omega_0 t\right\}$$

$$x(t+T/2) \neq -x(t)$$

Quadratic Anharmonic Oscillator

$F = -kx - \beta x^2 \qquad F = -\frac{\partial U}{\partial x} \Rightarrow U(x) = \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3$

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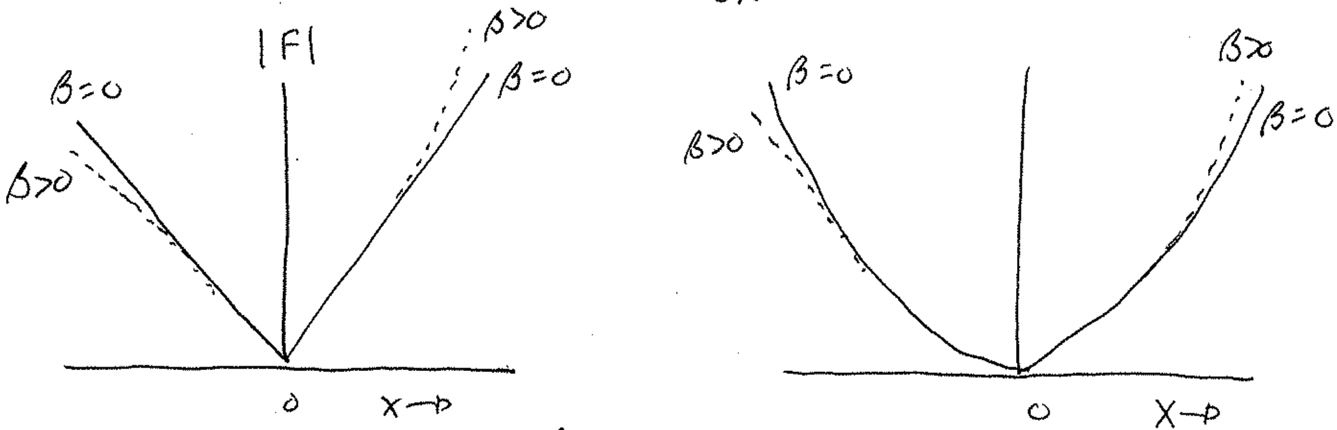


Asymmetric Restoring force

Asymmetric potential

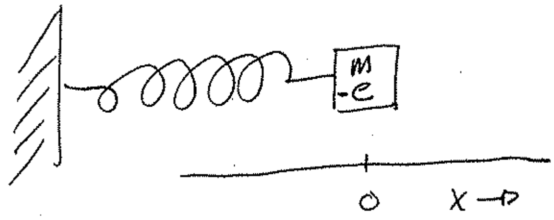
$\beta > 0 \Leftrightarrow$ spring stiffens in $z>0$ direction, softens in $z<0$ direction

Free oscillator: $\omega \simeq \left(1 - \left(\frac{\beta}{k}A\right)^2\right)\omega_0 \simeq \omega_0$ to second order in $\frac{\beta A}{k}$.

Note that the oscillator will spend more of it's time in the region $x<0$, the softer side of the potential. The average position is actually $\simeq -\frac{1}{2}\frac{\beta}{k}x_0^2 \equiv \delta$

$x(t) = \delta + A_1 \sin \omega t + A_2 \sin 2\omega t + A_4 \sin 4\omega t + \ldots$

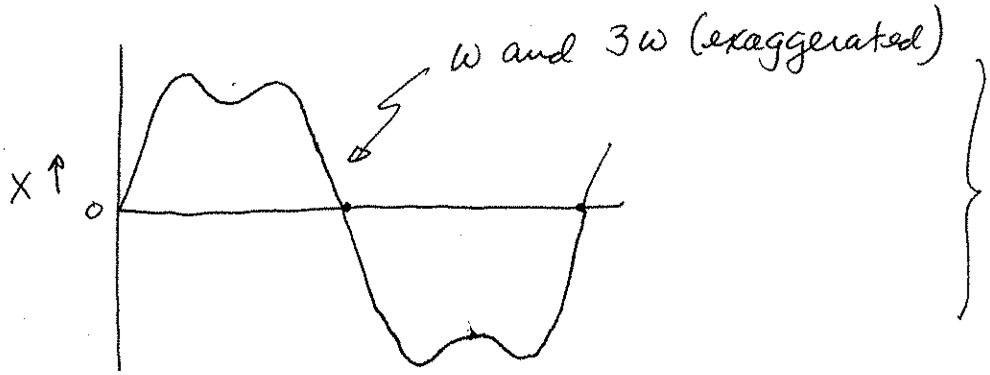
Anharmonic oscillators driven at a frequency $\omega \ll \omega_0$.



Applied force is $\vec{F} = -e\vec{E}(t)$

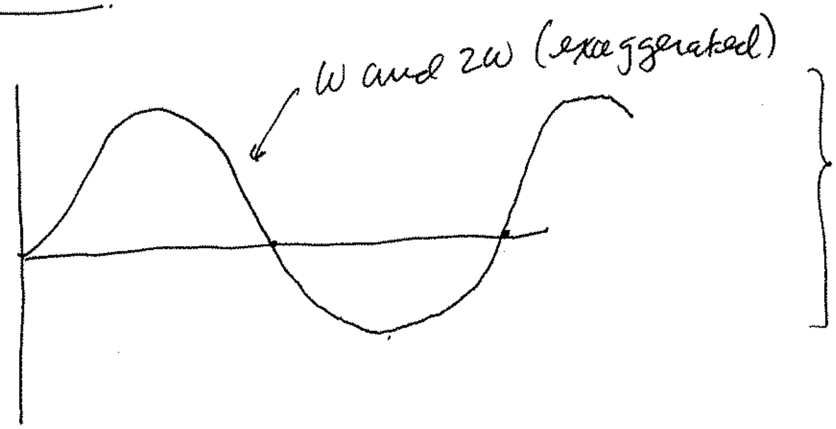
$$\vec{E}(t) = \vec{\mathcal{E}} e^{i\omega t} = \mathcal{E}\hat{x} e^{i\omega t}$$

Cubic



Symmetric response

quadratic



Asymmetric Response

In matter, model electron response using this anharmonic spring. The electrons are driven at ω, and dipoles oscillating at ω and $n\omega$ are created. These radiate at ω, which results in wave propagation, and at $n\omega$. The source polarization $\vec{P}(n\omega)$ is just the dipole moment density.

(24) Driven cubic anharmonic oscillator (electron on a spring) $\omega \ll \omega_0$

$$m\ddot{x} = -kx - \alpha x^3 - eE(t) - \gamma \dot{x} \quad , \quad E(t) = \mathcal{E} e^{i\omega t}$$

Assume that $x(t) = \sum_{n=0}^{\infty} A_n \sin((2n+1)\omega t)$, $\omega_0 = \sqrt{\frac{k}{m}}$

Simplify: $x(t) = A_1 e^{i\omega t} + A_3 e^{i3\omega t}$, $A_1 \gg A_3$

$$\dot{x}(t) = i\omega A_1 e^{i\omega t} + i3\omega A_3 e^{i3\omega t}$$

$$\ddot{x}(t) = -\omega^2 A_1 e^{i\omega t} - 9\omega^2 A_3 e^{i3\omega t}$$

$$x^3(t) \simeq A_1^3 e^{i3\omega t} \quad \text{(leading term)}$$

$e^{i\omega t}$ component: $-m\omega^2 A_1 = -kA_1 - e\mathcal{E} - i\omega\gamma A_1$

$e^{i3\omega t}$ component: $-9m\omega^2 A_3 = -kA_3 - \alpha A_1^3 - i3\omega\gamma A_3$

So, $A_1 = \dfrac{e\mathcal{E}}{m\omega^2 - k - i\omega\gamma} = \dfrac{e\mathcal{E}}{m(\omega^2 - \omega_0^2 - i\omega\gamma/m)}$

and $A_3 = \dfrac{\alpha A_1^3}{9m\omega^2 - k - i3\omega\gamma} = \dfrac{\alpha A_1^3}{m(9\omega^2 - \omega_0^2 - i3\omega\gamma/m)}$

Assume $X(t) = A_1 e^{i\omega t} + A_2 e^{i2\omega t}$, $A_1 >> A_2$

$$\dot{X} = i\omega A_1 e^{i\omega t} + i\, 2\omega A_2 e^{i2\omega t}$$

$$\ddot{X} = -\omega^2 A_1 e^{i\omega t} - 4\omega^2 A_2 e^{i2\omega t}$$

$$X^3 \triangleq A_1^3 e^{i3\omega t} \quad \text{(leading term)}$$

$e^{i\omega t}$ component: $-m\omega^2 A_1 = -k A_1 - \alpha \phi - e\mathcal{E} - i\omega\gamma A_1$

$e^{i2\omega t}$ component: $-m4\omega^2 A_2 = -k A_2 - \alpha\phi - \phi - i2\omega\gamma A_2$

ω: $A_1 = \dfrac{e\mathcal{E}}{m\omega^2 - k - i\omega\gamma}$

2ω: $A_2 (m4\omega^2 - k - i2\omega\gamma) = \phi \Rightarrow A_2 = 0$

Conclusion: anharmonic but symmetric potential for an electron can yield the "odd harmonics" of the driving frequency.

Also, $A_3 \propto \mathcal{E}^3$.

Optical third, fifth, seventh, ... harmonic generation can occur in centrosymmetric media.

Driven quadratic anharmonic oscillator ($\omega << \omega_0$)

$$m\ddot{x} = -kx - \beta x^2 - eE - \gamma\dot{x} \quad , \quad E(t) = \mathcal{E}e^{i\omega t}$$

Assume $x(t) = A_1 e^{i\omega t} + A_2 e^{i2\omega t} + C$

$$\dot{x} = i\omega A_1 e^{i\omega t} + i2\omega A_2 e^{i2\omega t}$$

$$\ddot{x} = -\omega^2 A_1 e^{i\omega t} - 4\omega^2 A_2 e^{i2\omega t}$$

$$x^2 \simeq A_1^2 e^{i2\omega t} \text{ (leading term)}$$

$e^{i\omega t}$ component: $-m\omega^2 A_1 = -kA_1 - \beta\phi - e\mathcal{E} - i\omega A_1\gamma$

$e^{i2\omega t}$ component: $-4m\omega^2 A_2 = -kA_2 - \beta A_1^2 - \phi - i2\omega\gamma A_2$

$$\omega: \quad A_1 = \frac{e\mathcal{E}}{m(\omega^2 - \frac{k}{m} - i\frac{\omega\gamma}{m})} = \frac{e\mathcal{E}}{m(\omega^2 - \omega_0^2 - i\omega\gamma/m)}$$

$$2\omega: \quad A_2 = \frac{\beta A_1^2}{(m4\omega^2 - k - i2\omega\gamma)} = \frac{\beta A_1^2}{m(4\omega^2 - \omega_0^2 - i\omega\gamma/m)}$$

If you assume $x(t) = A_1 e^{i\omega t} + A_3 e^{i3\omega t}$, you would find that $A_3 = \phi$.

Conclusion: anharmonic, asymmetric potentials can yield the "even harmonics" of the driving frequency.

Also, $A_2 \propto \mathcal{E}^2$, $A_4 \propto \mathcal{E}^4$, etc.

Optical second, fourth, ... harmonic generation occurs only in materials which are noncentrosymmetric.