

Characterizations of hyperbolic geometry among Hilbert geometries: A survey

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Abstract. This chapter is a survey of different approaches of characterizations of hyperbolic geometry among Hilbert geometries.

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1 Introduction

The Hilbert metric is a canonical metric associated to an arbitrary bounded convex domain. It was introduced by David Hilbert in 1894 as an example of a

metric for which the Euclidean straight lines are geodesics. Hilbert geometry generalizes Klein's model of hyperbolic geometry.

Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$). The *Hilbert metric* d_K on K is defined as follows. For any $x \in K$, let $d_K(x, x) = 0$. For distinct points x, y in K , assume the straight line passing through x, y intersects the boundary $\partial\bar{K}$ at two points a, b such that the order of these four points on the line is a, x, y, b as in Figure 1.

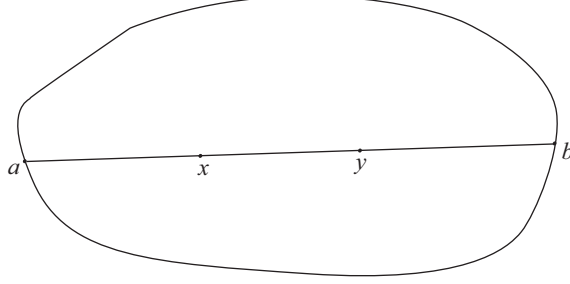


Figure 1. Hilbert metric

Denote the cross-ratio of the points by

$$[x, y, b, a] = \frac{\|b - x\|}{\|b - y\|} \cdot \frac{\|a - y\|}{\|a - x\|}$$

where $\|\cdot\|$ is the Euclidean norm of \mathbb{R}^n . Then the Hilbert metric is

$$d_K(x, y) = \frac{1}{2} \ln[a, x, y, b].$$

The metric space (K, d_K) is called a *Hilbert geometry*. Note that a Euclidean straight line in K is a geodesic under the metric d_K . When K is the unit open ball

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\},$$

(K, d_K) is the Klein's model of hyperbolic geometry.

Since the cross-ratio is invariant under a *perspectivity* P of center $O \in \mathbb{R}^n \cup \{\infty\}$, (K, d_K) and $(P(K), d_{P(K)})$ are isometric as Hilbert geometries. In particular, when K is an open ellipsoid (ellipse when $n=2$),

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{x_i^2}{a_i^2} < 1\}$$

for some nonzero numbers a_1, \dots, a_n , (K, d_K) is isometric to hyperbolic space.

Hilbert geometry has been studied under different viewpoints: affine geometry, Finsler geometry, dynamical system, etc. For the recent research activity on Hilbert geometry, see, for example, the papers by Colbois, Vernicos and Verovic [11, 33, 13], Förtsch, Karlsson and Noskov [15, 22], de la Harpe [19], Benoist [2, 3, 4, 5], Papadopoulos and Troyanov [27, 28, 29], Socié-Méthou [31, 32] and the book by Chern and Shen [9] and various chapters in the present handbook.

Hilbert geometry is such a rich subject. In this chapter, we focus on just one problem: to characterize hyperbolic geometry among Hilbert geometries. This is equivalent to projectively characterizing ellipsoids among open convex sets using intrinsic properties of Hilbert metrics.

The problem of characterization is studied through different approaches. In this chapter, we survey seven approaches and include one conjecture for this problem. It is interesting to find the subtle relationships between these different approaches. Since there exist characterizations of ellipsoids without using Hilbert metrics, it is also interesting to find relationships between the characterizations using Hilbert metrics and those which are not using Hilbert metrics.

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2 Reflections

A *reflection* in Hilbert or in Minkowski geometry is an isometric involution fixing a hyperplane. In hyperbolic geometry, there is a reflection through every hyperplane. Busemann and Kelly [8] characterized hyperbolic geometry among Hilbert geometries using this properties. More precisely:

Theorem 2.1 (Busemann–Kelly). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with a Hilbert metric d_K . Reflections in all the hyperplanes in K through one fixed point exist if and only if K is an ellipsoid.*

This result is proved on page 163 of [8] for the 2-dimensional case and on page 297 for the 3-dimensional case. The arguments extend to all dimensions.

3 Perpendicularity

Let us recall the definition of perpendicularity on page 121 of Busemann and Kelly's book [8].

If p and ξ are any point and line respectively in a metric space M , then a point f on ξ is a "foot of p on ξ " if

$$d_K(p, f) \leq d_K(p, x)$$

for all points x on ξ . A line η , intersecting ξ , is *perpendicular* to ξ if every point on η has the point of intersection of ξ and η as a foot on ξ . With this definition, the fact that η is perpendicular to ξ does not necessarily imply that ξ is perpendicular to η . If it does, the metric space M is said to have the *symmetry of perpendicularity*.

Kelly and Paige [24] showed that the symmetry of perpendicularity almost characterizes hyperbolic geometry among Hilbert geometries.

Theorem 3.1 (Kelly–Paige). *Let K be a bounded open convex set in \mathbb{R}^2 with a Hilbert metric d_K . (K, d_K) has the symmetry of perpendicularity if and only if K is an ellipse or the interior of a triangle. Therefore, if the boundary $\partial\bar{K}$ contains at most one line segment, then (K, d_K) has the symmetry of perpendicularity if and only if K is an ellipse.*

The proof of this result depends on the following characterization of an ellipse.

Lemma 3.2 (Kubota [26]). *Let Γ be a simple closed convex curve in the Euclidean plane and v an arbitrary direction. Let l_v be the line joining the two contact points of two tangents to Γ in the direction v . If for any v the line l_v cuts all chords of Γ in the direction v into two equal parts, then Γ is the boundary of an ellipse.*

Theorem 3.1 is easily generalized to higher dimensions. In higher dimensions, perpendicularity refers to lines in the same plane.

Theorem 3.3 (Kelly–Paige). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) such that $\partial\bar{K}$ is strictly convex. Let d_K be the Hilbert metric on K . (K, d_K) has the symmetry of perpendicularity if and only if K is an ellipsoid.*

In fact, if (K, d_K) has the symmetry of perpendicularity, any plane section of K has the symmetry of perpendicularity. Thus by Theorem 3.1 any plane section of K is an ellipse. Hence K is an ellipsoid due to the following lemma.

Lemma 3.4. *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$). Then the following assertions are equivalent:*

- (i) K is an ellipsoid.
- (ii) For each 2-dimensional plane H which meets the interior of K , the intersection $K \cap H$ is an ellipse.

This lemma is a weak version of Lemma 12.1 on page 226 of [17].

4 Ptolemaic inequality

Let M be a metric space. For points x and y in M , let xy denote the distance between x and y . The metric space M is called *Ptolemaic* if for each quadruple of points x_1, x_2, x_3, x_4 in M the *Ptolemaic inequality*

$$x_1x_2 \cdot x_3x_4 + x_1x_4 \cdot x_2x_3 \geq x_1x_3 \cdot x_2x_4$$

holds. If the inequality holds only in some neighborhood of each point, the space M is called *locally Ptolemaic*. Kay [23] proved that hyperbolic geometry is Ptolemaic and using this property to characterize hyperbolic geometry among Hilbert geometries.

Theorem 4.1 (Kay). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with a Hilbert metric d_K . K is an ellipsoid if and only if K is locally Ptolemaic.*

The idea of the proof is as follows.

Definition 4.2. The metric d_K is associated to a *Finsler metric* F_K on K as follows. For any $p \in K$ and $v \in T_pK = \mathbb{R}^n$,

$$F_K(p, v) := \frac{1}{2} \|v\| \left(\frac{1}{\|p - p^-\|} + \frac{1}{\|p - p^+\|} \right)$$

where p^- (respectively p^+) is the intersection point of the half line $p + \mathbb{R}^-v$ (respectively $p + \mathbb{R}^+v$) with ∂K and $\|\cdot\|$ is the Euclidean norm of \mathbb{R}^n .

Lemma 4.3 (Kay). *If a Hilbert metric is local Ptolemaic, then its tangent space at any point is also Ptolemaic.*

Lemma 4.4 (Schoenberg [30]). *A Finsler metric is Riemannian (the unit sphere of its tangent space is an ellipsoid) if and only if it is Ptolemaic.*

Lemma 4.5 (Kay, Theorem 2 in [23]). *Perpendicularity under a Hilbert metric coincides with perpendicularity under its associated Finsler metric.*

Therefore Theorem 4.1 is implied by Theorem 3.1 (Kelly-Paige): if (K, d_K) has the symmetry of perpendicularity, K is an ellipsoid.

As a corollary, we obtain the following basic fact in Hilbert geometry:

Corollary 4.6 (page 296 in [23]). *A Hilbert geometry is Riemannian if and only if it is hyperbolic.*

5 Curvature

In a Hilbert geometry, the curvature sign at a point is defined in a qualitative rather than a quantitative sense.

Definition 5.1. Let K be a bounded open convex set in \mathbb{R}^n ($(n \geq 2)$) with a Hilbert metric d_K . The curvature at $p \in K$ is respectively non-negative or non-positive *in the sense of Busemann* if there exists a neighborhood U of p such that for every x, y in U we have

$$2d_K(\bar{x}, \bar{y}) \geq d_K(x, y),$$

respectively

$$2d_K(\bar{x}, \bar{y}) \leq d_K(x, y),$$

where \bar{x}, \bar{y} are the midpoints respectively of the Euclidean line segments from p to x and p to y under d_K .

The curvature is said to be *determinate* at a point if it is either positive or negative at that point. The curvature is said to be *indeterminate* at a point if it is neither positive nor negative.

Kelly and Straus [25] characterized hyperbolic geometry using curvature of Hilbert geometries.

First, it is established in the 2-dimensional case.

Theorem 5.2 (Kelly–Straus). *Let K be a bounded open convex set in \mathbb{R}^2 with a Hilbert metric d_K . K is an ellipse if and only if the curvature is determinate everywhere in K .*

Since hyperbolic space has negative curvature, this Theorem implies in particular that if K has everywhere determinate curvature, then this curvature is non-positive.

The proof uses the properties of a projective center of K .

Consider the projective space $P^2(\mathbb{R})$ as \mathbb{R}^2 plus ∞ . Then K can be considered in $P^2(\mathbb{R})$. A point p is called a *projective center* of K if there is a projective transformation $f : P^2(\mathbb{R}) \rightarrow P^2(\mathbb{R})$ such that $f(K) \subset \mathbb{R}^2$ is a centrally symmetric convex domain with center $f(p)$.

Lemma 5.3 (Kelly–Straus). *If p is a point of determinate curvature then it is a projective center.*

Therefore Theorem 5.2 is true due to the following characterization of an ellipse.

Lemma 5.4 (Kajima [21]). *Let Γ be a simple closed convex curve in \mathbb{R}^2 . If every point in Γ is a projective center, then Γ is the boundary of an ellipse.*

Theorem 5.2 is easily generalized to higher dimensions.

Theorem 5.5 (Kelly–Straus). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with a Hilbert metric d_K . K is an ellipsoid if and only if the curvature is determinate everywhere in K .*

In fact, if the curvature of (K, d_K) is determinate everywhere, the curvature of any plane section of K is determinate everywhere. Thus by Theorem 5.2 any plane section of K is an ellipse. Hence K is an ellipsoid due to Lemma 3.4.

As a corollary, we get Egloff’s result [14]:

Corollary 5.6. *A Hilbert geometry (K, d_K) satisfies the $CAT(0)$ condition if and only if K is an ellipsoid.*

For the notion of $CAT(0)$ metric spaces, see, for example, the book [7]. This corollary follows from Theorem 5.5 since every geodesic metric space satisfying $CAT(0)$ has non-positive curvature in the sense of Busemann.

6 Median

The following is a well-known fact in constant sectional curvature geometry (see [16] Chapter 7, especially problem K-19):

The three medians of any triangle pass through one point.

Guo [18] proved that this property characterizes hyperbolic geometry among Hilbert geometries.

Theorem 6.1. *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with a Hilbert metric d_K . K is an ellipsoid if and only if the three medians of any triangle in (K, d_k) pass through one point.*

To prove the result, it is enough to verify it for $n = 2$.

One idea used in the proof is Fritz John’s ellipse [20].

Lemma 6.2 (John). *Any bounded open convex set K in \mathbb{R}^2 contains a unique ellipse E with maximal Euclidean area. Furthermore $\partial\overline{K} \cap \partial\overline{E}$ contains at least three points.*

7 Isometry group

Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with associated Hilbert metric d_K . Denote by $\text{Isom}(K, d_k)$ the isometry group of (K, d_K) . The action of a discrete subgroup Γ of $\text{Isom}(K, d_k)$ on K is called *proper* if the quotient topological space K/Γ is Hausdorff.

In 1960, Benzécri [1] characterized hyperbolic geometry using properties of the action of discrete subgroups of $\text{Isom}(K, d_k)$.

Theorem 7.1 (Benzécri). *Assume $\partial\bar{K}$ is strictly convex. If (K, d_K) admits a proper action of a discrete subgroup of $\text{Isom}(K, d_k)$ such that the quotient is compact, then K is an ellipsoid.*

The condition that $\partial\bar{K}$ is strictly convex is necessary. For example, if K is a triangle, there exist discrete subgroups of $\text{Isom}(K, d_k)$ whose actions on K are proper and the quotients are compact.

We also quote the following generalization of Benzécri's result.

Theorem 7.2 (Benoist, Proposition 6.2 in [2]). *If the support of the curvature on $\partial\bar{K}$ has positive measure, then (K, d_K) admits a proper action of a discrete subgroup of $\text{Isom}(K, d_k)$ such that the quotient is compact if and only if K is an ellipsoid.*

Colbois and Verovic [10] obtained the following generalization of Benzécri's result in the case of convex domains with smooth boundaries.

Theorem 7.3 (Colbois–Verovic). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) whose boundary $\partial\bar{K}$ is a hypersurface of class C^3 which is strictly convex (in the sense that the Hessian is positive definite). Then if $\partial\bar{K}$ is not an ellipsoid, any discrete subgroup of the isometry group $\text{Isom}(K, d_k)$ whose action on K is proper is finite.*

Recall that $F_K(p, v)$ is the associated Finsler metric (Definition 4.2). To any $p \in K$, let

$$B_K(p) = \{v \in \mathbb{R}^n \mid F_K(p, v) \leq 1\}$$

be the open unit ball in the tangent space $T_p K$. Let ω_n be the Euclidean volume of the open unit ball of the standard Euclidean space \mathbb{R}^n . The *density function* $\sigma : K \rightarrow \mathbb{R}$ is defined by

$$\sigma(p) = \frac{\omega_n}{\text{Vol}(B_K(p))},$$

where Vol is the canonical Lebesgue measure of \mathbb{R}^n .

Definition 7.4. The *Hilbert measure* on K is defined as

$$\mu_K(A) = \int_A \sigma(p) dp$$

for any Borel set A of K .

Corollary 7.5 (Colbois–Verovic). *Let K be as above. If $\partial\overline{K}$ is not an ellipsoid, then (K, d_K) does not allow quotients of finite volume by discrete subgroups of $\text{Isom}(K, d_K)$ whose action on K is proper.*

The idea of the proof of Theorem 7.3 is as follows. First, if an infinite group of $\text{Isom}(K, d_K)$ had a proper action on K , then every point of K is sent to $\partial\overline{K}$. Then, the Finsler metric $F_K(p)$ is approaching a Riemannian metric when the point p is approaching $\partial\overline{K}$. Thus F_K is identified as the Riemannian metric which is the limit of a sequence of Riemannian metrics. To continue the proof of Theorem 7.3, the following lemma is applied (see, for example, Busemann [6], page 85).

Lemma 7.6 (Beltrami, 1866). *Let a connected open set X of the projective space $P(\mathbb{R}^{n+1})$ be metrised so that the metric is Riemannian and the geodesics lie on projective lines. Then the sectional curvature of this Riemannian metric is constant.*

Since (K, d_K) is non compact, the curvature of d_K is non-positive. By a theorem of Busemann ([6], page 269), the metric space (K, d_K) has non-positive curvature in the sense of Busemann (see Definition 5.1). Theorem 5.2 (Kelly–Straus) implies that K is an ellipsoid.

8 Ideal triangles

A triangle in K has three vertices in K with any two vertices joined by a geodesic. When the three vertices of a triangle in K are on $\partial\overline{K}$, it becomes an *ideal triangle* in K .

In hyperbolic geometry, the area of any ideal triangle is a constant: π .

In a Hilbert geometry, the area of an ideal triangle is calculated using the Hilbert measure (see Definition 7.4). In [12], the authors characterized an ellipsoid using the area of ideal triangles.

Theorem 8.1 (Colbois–Vernicos–Verovic). *Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with a Hilbert metric d_K . If the area of every ideal triangle in (K, d_K) is a constant, K is an ellipsoid.*

Fritz John’s ellipse, Lemma 6.2, is also used in the proof.

9 Spectrum: a conjecture

Let K be a bounded open convex set in \mathbb{R}^n ($n \geq 2$) with associated Hilbert metric d_K . Recall that $F_K(p, v)$ is the associated Finsler metric of (K, d_K) (see Definition 4.2). Let l_p be a linear form on the tangent space $T_p K$. Define

$$\|l_p\|_K^* = \sup\{l_p(v) : v \in T_p K, F_K(p, v) = 1\}.$$

Recall that μ_K is the Hilbert measure on K (see Definition 7.4).

Using the Raleigh quotients, the *bottom of the spectrum* of K is defined as

$$\lambda_1(K) = \inf \frac{\int_K \|df_p\|_K^{*2} d\mu_K(p)}{\int_K f^2(p) d\mu_K(p)}$$

where the infimum is taken over all non zero lipschitz function with compact support in K .

Theorem 9.1 (Vernicos [34]).

$$\lambda_1(K) \leq \frac{(n-1)^2}{4}.$$

It is natural to ask whether the bottom of the spectrum can characterize hyperbolic geometry. The following conjecture is due to B. Colbois.

Conjecture. *If $\lambda_1(K) = \frac{(n-1)^2}{4}$, then K is an ellipsoid.*

References

- [1] J.-P. Benzécri, Sur les variétés localement affines et projectives. *Bull. Soc. Math. France* 88 (1960), 229–332.
- [2] Y. Benoist, Convexes divisibles I. In *Algebraic groups and arithmetic*, Tata Inst. Fund. Res., Mumbai, 2004, 339-374.
- [3] Y. Benoist, Convexes divisibles II. *Duke Math. J.* 120 (2003), 97-120.
- [4] Y. Benoist, Convexes divisibles III. *Ann. Sci. Ec. Norm. Supér.* (4) 38 (2005), 793-832.
- [5] Y. Benoist, Convexes divisibles IV. Structure du bord en dimension 3. *Invent. Math.* 164 (2006), 249-278.
- [6] H. Busemann, *The Geometry of Geodesics*. Academic Press, New York, 1955.
- [7] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*. Springer-Verlag Berlin Heidelberg 1999.
- [8] H. Busemann and P. J. Kelly, *Projective Geometry and Projective metrics*. Academic Press, New York, 1953.

- [9] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*. Nankai Tracts in Mathematics, vol. 6 World Scientific 2005.
- [10] B. Colbois and P. Verovic, Rigidity of Hilbert metrics. *Bull. Austral. Math. Soc.* 65 (2002), 23–34.
- [11] B. Colbois and P. Verovic, Hilbert geometry for strictly convex domains. *Geom. Dedicata* 105 (2004), 29–42.
- [12] B. Colbois, C. Vernicos, and P. Verovic, L’aire des triangles idéaux en géométrie de Hilbert. *Enseign. Math. (2)* 50 (2004), 203–237.
- [13] B. Colbois, C. Vernicos, and P. Verovic, Hilbert geometry for convex polygonal domains. *J. Geom.* 100 (2011), 37–64.
- [14] D. Eglhoff, Uniform Finsler Hadamard manifolds. *Ann. Inst. H. Poincaré Phys. Théor.* 66 (1997), no. 3, 323–357.
- [15] T. Förtsch and A. Karlsson, Hilbert metrics and Minkowski norms. *J. Geom.* 83 (2005), 22–31.
- [16] M. Greenberg, *Euclidean and Non-Euclidean Geometry*. 2nd Ed., San Francisco: Freeman, 1980.
- [17] P. M. Gruber, *Convex and Discrete Geometry*. Springer-Verlag Berlin Heidelberg 2007.
- [18] R. Guo, A characterization of hyperbolic geometry among Hilbert geometry. *J. Geom.* 89 (2008), 48–52.
- [19] P. de la Harpe, On Hilbert’s metric for simplices, *Geometric group theory*. Vol. 1, 97–119. Lond. Math. Soc. Lect. Note Ser. **181**, Cambridge Univ. Press, Cambridge, 1993.
- [20] F. John, Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, Interscience Publishers, Inc., New York, N. Y., 1948, 187–204.
- [21] T. Kajima, On a characteristic property of the conic and the quadric. *Sci. Rep.*, Tohoku Univ. 8, 1919
- [22] A. Karlsson and G. A. Noskov, The Hilbert metric and Gromov hyperbolicity. *Enseign. Math. (2)* 48 (2002), 73–89.
- [23] D. Kay, The ptolemaic inequality in Hilbert geometry. *Pacific J. Math.* 21 (1967), 293–301.
- [24] P. J. Kelly and L. J. Paige, Symmetric perpendicularity in Hilbert geometry. *Pacific J. Math.* 2 (1952), 319–322.
- [25] P. J. Kelly and E. Straus, Curvature in Hilbert geometries. *Pacific J. Math.* 8 (1958), 119–125.
- [26] T. Kubota, On a characteristic property of the ellipse. *Tohoku Math. J.* 9 (1916), 148–151.
- [27] A. Papadopoulos and M. Troyanov, Weak metrics on Euclidean domains. *JP J. Geom. Topol.* 7 (2007), 23–43.

- [28] A. Papadopoulos and M. Troyanov, Weak Finsler structures and the Funk weak metric. *Math. Proc. Cambridge Philos. Soc.* 147 (2009), 419-437.
- [29] A. Papadopoulos and M. Troyanov, Harmonic symmetrization of convex sets and of Finsler structures, with applications to Hilbert geometry. *Expo. Math.* 27 (2009), 109-124.
- [30] I. J. Schoenberg, A Remark on M. M. Day's Characterization of Inner-Product Spaces and a Conjecture of L. M. Blumenthal. *Proc. Amer. Math. Soc.* 3 (1952), 961-964.
- [31] E. Socié-Méthou, *Comportements asymptotiques et rigidités des géométries de Hilbert*. PhD thesis, University of Strasbourg, 2000.
- [32] E. Socié-Méthou, Behaviour of distance functions in Hilbert-Finsler geometry. *Differential Geom. Appl.* 20 (2004), 1-10.
- [33] C. Vernicos, Introduction aux géométries de Hilbert. In *Actes de Séminaire de Théorie Spectrale et Géométrie*. Vol. 23. Année 2004-2005, Sémin. Théor. Spectr. Géom., 23, Univ. Grenoble I, Saint-Martin-d'Hères, 2005, 145-168.
- [34] C. Vernicos, Spectral radius and amenability in Hilbert geometries. *Houston J. Math.* 35 (2009), 1143-1169.