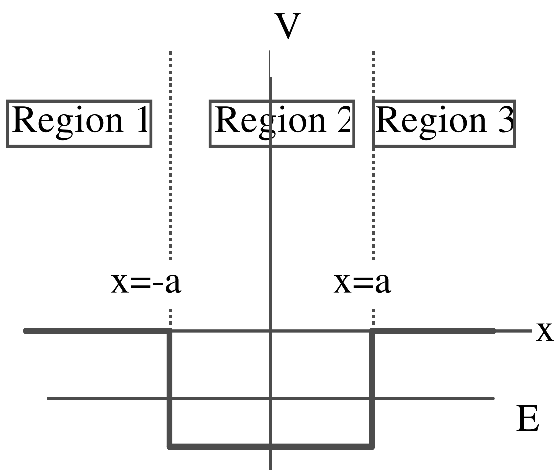
FINITE SQUARE WELL



x<-a REGION 1

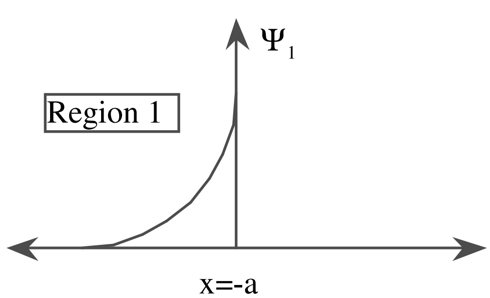
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1(x)+0.\psi_1(x)=E\psi_1(x) \qquad \frac{d^2}{dx^2}\psi_1(x)=-\frac{2mE}{\hbar^2}\psi_1(x)$$

$$\psi_1(x)=C'e^{i\sqrt{\frac{2mE}{\hbar^2}}x}+Ce^{-i\sqrt{\frac{2mE}{\hbar^2}}x}$$

Assume $E<0$ *i.e.* $E=-|E|$

$$C'=0 \text{ otherwise } \psi_1(x\to-\infty)=C'e^{i.i\sqrt{\frac{2m|E|}{\hbar^2}}(-\infty)}=C'e^{\sqrt{\frac{2m|E|}{\hbar^2}}(\infty)}\to\infty$$

$$\psi_1(x)=Ce^{-i\sqrt{\frac{2mE}{\hbar^2}}x}\xrightarrow{E<0}Ce^{\sqrt{\frac{2m|E|}{\hbar^2}}x}\xrightarrow{Goswami}Ce^{\beta x} \qquad \beta=\sqrt{\frac{2m|E|}{\hbar^2}}$$



x>a REGION 3

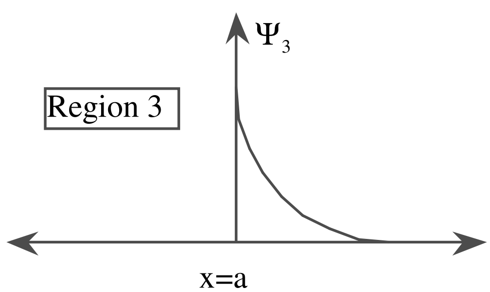
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3(x)+0.\psi_3(x)=E\psi_3(x) \qquad \frac{d^2}{dx^2}\psi_3(x)=-\frac{2mE}{\hbar^2}\psi_3(x)$$

$$\psi_3(x)=De^{i\sqrt{\frac{2mE}{\hbar^2}}x}+D'e^{-i\sqrt{\frac{2mE}{\hbar^2}}x}$$

For $E < 0$, *i.e.* $E = -|E|$

$D' = 0$ otherwise $\psi_3(x\to+\infty)=D'e^{-i.i\sqrt{\frac{2m|E|}{\hbar^2}}(+\infty)}=D'e^{+\sqrt{\frac{2m|E|}{\hbar^2}}(\infty)}\to\infty$

$$\psi_3(x)=De^{i\sqrt{\frac{2mE}{\hbar^2}}x}\xrightarrow{E<0}De^{-\sqrt{\frac{2m|E|}{\hbar^2}}x}\xrightarrow{Goswami}De^{-\beta x}$$



-a<x<a REGION 2

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2(x)+V(x)\psi_2(x)=E\psi_2(x) \qquad \frac{d^2}{dx^2}\psi_2(x)=-\frac{2m(E-V(x))}{\hbar^2}\psi_2(x)$$

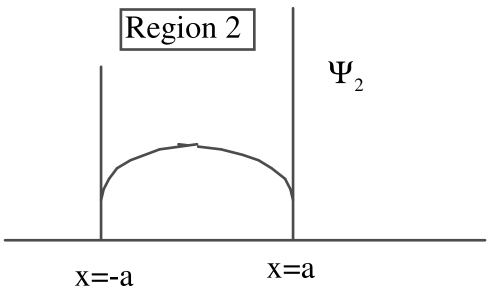
$$\psi_2(x)=A'e^{i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x}+B'e^{-i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x}$$

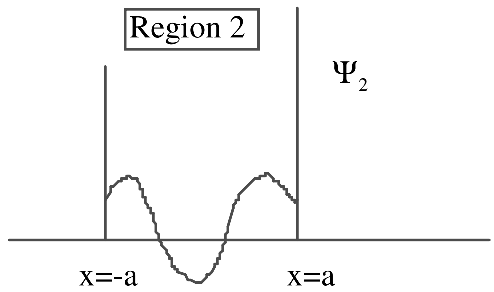
notice $E > V(x)$ for our case so solutions are oscillatory

Also $A' = \pm B'$ because of symmetry (general argument can be proved rigourously, but seems clear here because there is nothing in the potential to distinguish +x from -x)

Either (EVEN SOLUTION)

$$\psi_2(x) = A'\left(e^{i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x} + e^{-i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x} \right) = A\cos\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x\right)$$

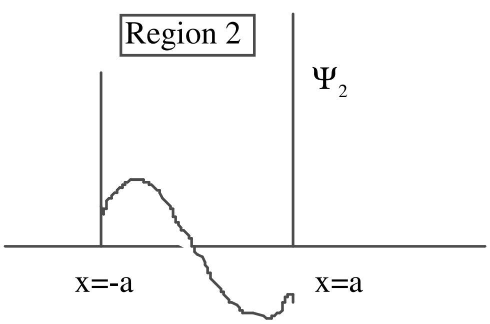


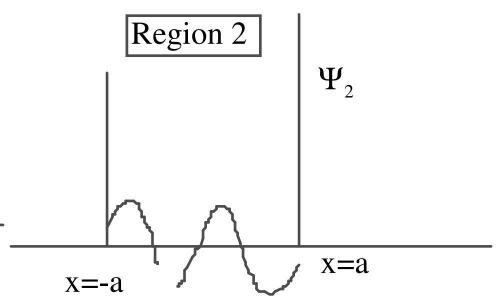


Notice that the cosine is not necessarily zero at the boundaries of region 2.

Or (ODD SOLUTION)

$$\psi_2(x) = A'\left(e^{i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x} - e^{-i\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x} \right) = B\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x\right)$$





Again, the sine functions are not necessarily zero at the boundaries of region 2.

We can also write

$$\psi_2(x) = A\cos\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x\right) + B\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}x\right)$$

expecting that when $A \neq 0$ then $B=0$ and when $B \neq 0$ then $A=0$. Now we have Goswami's notation. Actually, we don't need to invoke the symmetry argument, but simply let the matching of the boundary conditions bring out the sines and cosines for us, but it's nicer this way.

BOUNDARY CONDITIONS

Wavefunction continuous at x = -a

$$\psi_2(-a) - \psi_1(-a) = 0$$

$$A\cos\left(-\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) - B\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) - Ce^{-\beta a} = 0$$

Wavefunction derivative continuous at x = -a

$$\left.\frac{d\psi_2(x)}{dx}\right|_{x=-a} - \left.\frac{d\psi_1(x)}{dx}\right|_{x=-a} = 0$$

$$A\sqrt{\frac{2m(E-V(x))}{\hbar^2}}\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) + B\sqrt{\frac{2m(E-V(x))}{\hbar^2}}\cos\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) - \beta Ce^{-\beta a} = 0$$

Wavefunction continuous at $x = a$

$$\psi_2(a) - \psi_3(a) = 0$$

$$A\cos\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) + B\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) - De^{-\beta a} = 0$$

Wavefunction derivative continuous at $x = a$

$$\left.\frac{d\psi_2(x)}{dx}\right|_{x=a} - \left.\frac{d\psi_3(x)}{dx}\right|_{x=a} = 0$$

$$-A\sqrt{\frac{2m(E-V(x))}{\hbar^2}}\sin\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) + B\sqrt{\frac{2m(E-V(x))}{\hbar^2}}\cos\left(\sqrt{\frac{2m(E-V(x))}{\hbar^2}}a\right) + \beta De^{-\beta a} = 0$$

Write these in matrix form:

$$\begin{pmatrix} cos(k'a) & -sin(k'a) & -e^{-\beta a} & 0 \\ k'sin(k'a) & k'cos(k'a) & -\beta e^{-\beta a} & 0 \\ cos(k'a) & sin(k'a) & 0 & -e^{-\beta a} \\ -k'sin(k'a) & k'cos(k'a) & 0 & \beta e^{-\beta a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

$$where\ k' = \sqrt{\frac{2m(E - V(x))}{\hbar^2}} \xrightarrow{Goswami} \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$$

A solution exists for *A*, *B*, *C*, *D* if the determinant vanishes:

$$\begin{vmatrix} cos(k'a) & -sin(k'a) & -e^{-\beta a} & 0 \\ k'sin(k'a) & k'cos(k'a) & -\beta e^{-\beta a} & 0 \\ cos(k'a) & sin(k'a) & 0 & -e^{-\beta a} \\ -k'sin(k'a) & k'cos(k'a) & 0 & \beta e^{-\beta a} \end{vmatrix} = 0$$

which leads to

$$\left(\tan(k'a) - \frac{\beta}{k'}\right)\left(\tan(k'a) + \frac{k'}{\beta'}\right) = 0$$

Cannot have both brackets zero at same time, so we are led to two sets of solutions, which correspond to the previously mentioned condition that when $A \neq 0$ then $B = 0$ and when $B \neq 0$ then $A = 0$. It's not possible to find values for E (recall both k' and β depend on E) analytically - we have to resort to a graphical method, nicely outlined in Goswami. (Continue with Goswami 4.32)