IS there an easier way??

Henceforth $K \to \alpha$ (redefine spring constant)

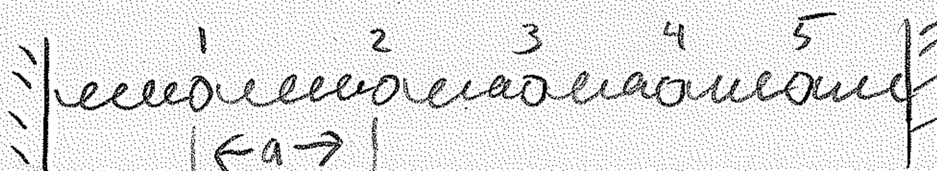
$k \equiv 2\pi/\lambda$, where λ is the wavelength of the desired normal mode.

<u>Ansatz #2</u>: for any periodic system we claim that a given mass n will oscillate as:

$$\boxed{x_n(t) = A \sin(\omega_K t) \sin(kna)}$$

$\hookrightarrow$ independent of n!

<u>Example</u>: find the natural frequencies of 5 coupled masses.



What is the longest wavelength possible?

$$\lambda = 12a \implies k = 2\pi/12a = \pi/6a$$

Step ① coupled ODEs

$$m\ddot{x}_1 = -\alpha x_1 - \alpha(x_1 - x_2)$$
$$m\ddot{x}_2 = -\alpha(x_2 - x_1) - \alpha(x_2 - x_3)$$
$$m\ddot{x}_3 = -\alpha(x_3 - x_2) - \alpha(x_3 - x_4)$$
$$m\ddot{x}_4 = \ldots$$
$$m\ddot{x}_5 = \ldots$$

Step ② $k = \pi/6a$, $x_n(t) = A\sin(kna)\sin(\omega_k t)$

$k_1 = \pi/6a$, $k_2 = \pi/3a$, $k_3 = \pi/2a$, $k_4 = \frac{2\pi}{3a}$, $k_5 = \frac{5\pi}{6a}$

$n=1$ mass $\Rightarrow x_1(t) = \frac{A}{2}\sin\omega_k t$

$n=2$ $\quad x_2(t) = \frac{\sqrt{3}}{2}A\sin\omega_k t$

$n=3$ $\quad x_3(t) = A\sin\omega_k t$

$n=4$ $\quad x_4(t) = \frac{\sqrt{3}}{2}A\sin\omega_k t$

$n=5$ $\quad x_5(t) = \frac{A}{2}\sin\omega_k t$

find ω_k by subbing $x_1(t)$ into 1st ODE:

i.e. $-m\omega_k^2 \frac{A}{2} = -\frac{A}{2}\alpha + A\left(\frac{\sqrt{3}}{2}\right)\alpha - \frac{A}{2}\alpha$

$= \alpha(\sqrt{3} - 2)$

or $\boxed{\omega_k = \sqrt{\alpha/m}\,\sqrt{2-\sqrt{3}}}$ natural frequency of the 1st normal mode

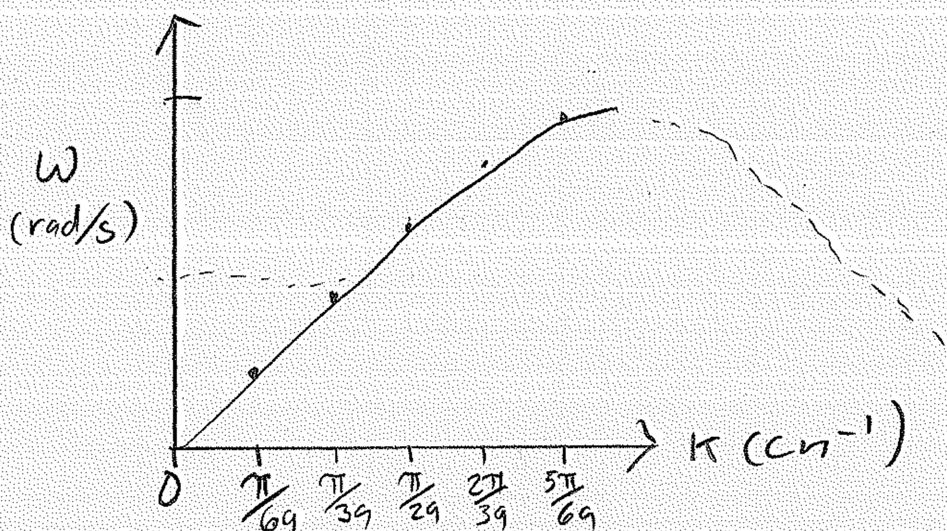
What is the frequency of the second normal mode??

First question: what is λ of the 2nd normal mode?

$\lambda = 6a \Rightarrow k = 2\pi/6a = \pi/3a$

PhET Lab:

Part 3: goal graph ω (rad/s) vs. k (cm^{-1})



$\omega \to 2\sqrt{\alpha/m}$
as $k \to \pi/a$

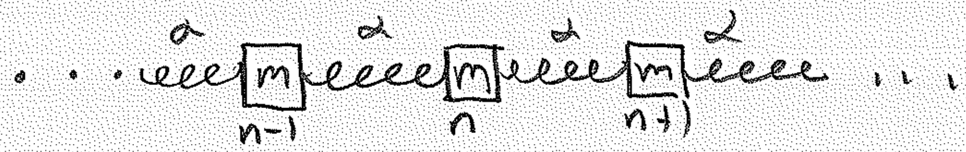
Now try 10 masses on your own.

Why do we converge to $2\sqrt{\alpha/m}$?

- the masses ultimately "see" just two time constants that each contribute a $\sqrt{\alpha/m}$ natural frequency component

i.e. if the spring-masses had a frequency faster than $2\sqrt{\alpha/m}$, the springs would have to break

Now consider a chain of infinite masses:



How many normal modes do expect?

Step (1) equations of motion (for the n^{th} mass)

$$m\ddot{x}_n = -\alpha(x_n - x_{n-1}) - \alpha(x_n - x_{n+1})$$
$$= -2\alpha x_n + \alpha x_{n-1} + \alpha x_{n+1}$$

Step (2) find natural frequencies

use ansatz #2: $x_n(t) = A\sin(\omega_k t)\sin(kna)$

OR

$x_n(t) = Ae^{i\omega t}e^{ikna}$

where $\omega_k \to \omega$ (continuous ω)

Sub in equation of motion (1):

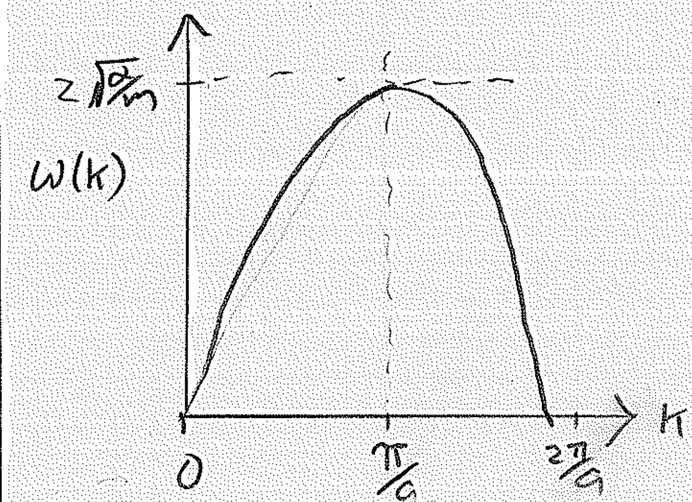
$$-m\omega^2 A e^{ikna} = -2\alpha e^{ikna} + \alpha e^{ik(n-1)a} + \alpha e^{ik(n+1)a}$$

$$\Rightarrow -m\omega^2 = \alpha(e^{-ika} + e^{ika} - 2)$$
$$= \alpha(2\cos(ka) - 2), \quad \cos 2x = \cos^2 x - \sin^2 x$$
$$= -2\alpha(2\sin^2(ka/2))$$

$$\boxed{\omega(k) = 2\sqrt{\alpha/m}\,\sin(ka/2)}$$ dispersion relation

k & ω are now continuous for this infinite system.

dispersion relation $\omega(k)$: gives all possible natural frequencies allowed for our 1D infinite system.



ω_D = Debye frequency

the highest natural frequency supported by a periodic system.

Homework #4. what is $\omega(k)$ for