Day 3 What are the osc amplitudes?

Double pendulum: $\omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} k/m + g/\ell & -k/m \\ -k/m & k/m + g/\ell \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$

$$\Rightarrow \lambda = g/\ell \quad \text{or} \quad \lambda = 2\tfrac{k}{m} + g/\ell$$

$$\omega_A = \sqrt{g/\ell} \qquad \omega_B = \sqrt{g/\ell + 2k/m}$$

What are the normal modes?

$\Rightarrow$ find eigen-vectors!

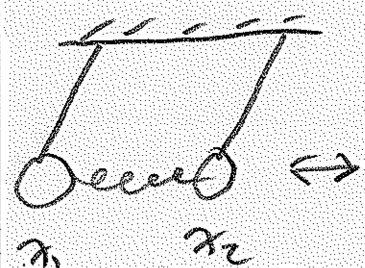
Case 1 $\lambda = g/\ell$

$$\Rightarrow 0 = \begin{bmatrix} k/m & -k/m \\ -k/m & k/m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\Rightarrow 0 = k/m\,(A_1 - A_2)$$

1st normal mode: $A_1 = A_2 \equiv A$

$$\left.\begin{aligned} x_1 &= A \sin \omega_A t \\ x_2 &= A \sin \omega_B t \end{aligned}\right\} \text{ spring is not stretched}$$



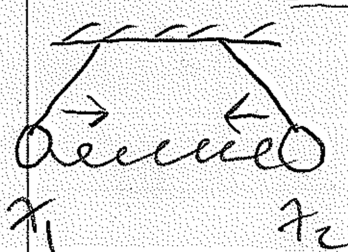
Case 2 $\lambda = 2k + mg/\ell$

$$\Rightarrow 0 = \begin{bmatrix} -k/m & -k/m \\ -k/m & -k/m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$0 = -k/m\,(A_1 + A_2)$$

2nd normal mode: $A_1 = -A_2$

$$x_1 = A \sin \omega_B t$$
$$x_2 = -A \sin \omega_B t$$

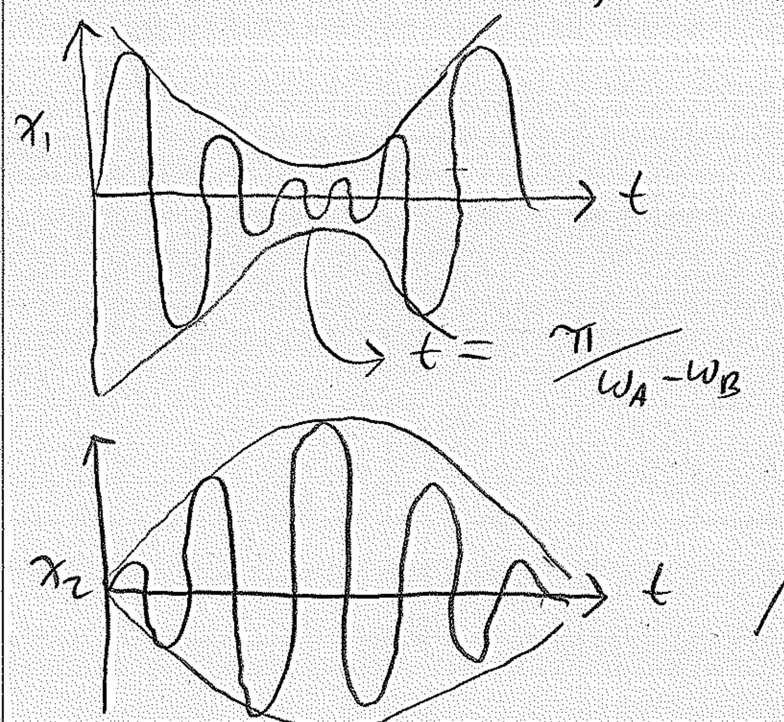


Superpositions of normal modes are also solutions

let's consider: $x_1 = C\sin\omega_A t + D\sin\omega_B t$

$$x_2 = C\sin\omega_A t - D\sin\omega_B t$$

lets graph when $C=D$, at $\omega_A t - \omega_B t = \pi$, x_2 moves alot, x_1 does not move



energy beats between the pendulums.

<u>Review</u>:

Step ① write equations of motion

Step ② find the natural frequencies (eigenvalues)

Step ③ solve for amplitudes & Normal modes (eigen vectors)

IS there an easier way??

Henceforth $K \to \alpha$ (redefine spring constant)

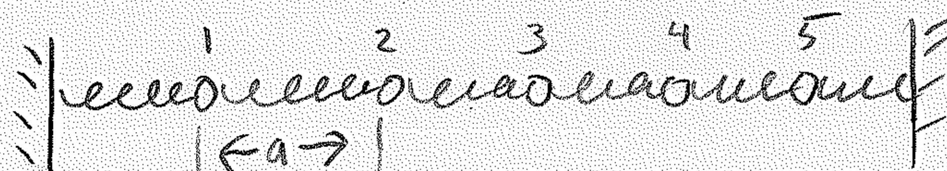
$k \equiv 2\pi/\lambda$, where λ is the wavelength of the desired normal mode.

Ansatz #2: for any periodic system we claim that a given mass n will oscillate as:

$$\boxed{x_n(t) = A \sin(\omega_k t) \sin(kna)}$$

independent of n!

Example: find the natural frequencies of 5 coupled masses.



What is the longest wavelength possible?

$$\lambda = 12a \implies k = 2\pi/12a = \pi/6a$$

Step ① coupled ODEs

$$m\ddot{x}_1 = -\alpha x_1 - \alpha(x_1 - x_2)$$
$$m\ddot{x}_2 = -\alpha(x_2 - x_1) - \alpha(x_2 - x_3)$$
$$m\ddot{x}_3 = -\alpha(x_3 - x_2) - \alpha(x_3 - x_4)$$
$$m\ddot{x}_4 = \ldots$$
$$m\ddot{x}_5 = \ldots$$

Step ② $k = \pi/6a$, $\quad x_n(t) = A\sin(kna)\sin(\omega_k t)$

$$k_1 = \frac{\pi}{6a}, \quad k_2 = \frac{\pi}{3a}, \quad k_3 = \frac{\pi}{2a}, \quad k_4 = \frac{2\pi}{3a}, \quad k_5 = \frac{5\pi}{6a}$$

$n=1$ mass $\Rightarrow x_1(t) = \frac{A}{2}\sin\omega_k t$

$n=2$ $\quad x_2(t) = \frac{\sqrt{3}}{2}A\sin\omega_k t$

$n=3$ $\quad x_3(t) = A\sin\omega_k t$

$n=4$ $\quad x_4(t) = \frac{\sqrt{3}}{2}A\sin\omega_k t$

$n=5$ $\quad x_5(t) = \frac{A}{2}\sin\omega_k t$

find ω_k by subbing $x_1(t)$ into 1st ODE:

i.e. $-m\omega_k^2\frac{A}{2} = -\frac{A}{2}\alpha + A\left(\frac{\sqrt{3}}{2}\right)\alpha - \frac{A}{2}\alpha$

$$= \alpha(\sqrt{3}-2)$$

or $\boxed{\omega_k = \sqrt{\alpha/m}\,\sqrt{2-\sqrt{3}}}$ natural frequency of the 1st normal mode

What is the frequency of the second normal mode??

First question: what is λ of the 2nd normal mode?

$\lambda = 6a \Rightarrow k = \frac{2\pi}{6a} = \frac{\pi}{3a}$