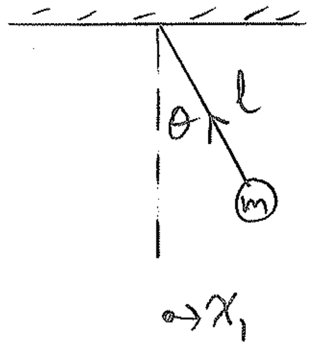
427 Topic #1 coupled oscillators. What is ω_0?

Consider 1 pendulum:



$F = ma \;\rightarrow\; \vec{\tau} = I\ddot{\theta}$

$-|\vec{r} \times \vec{F}| = m\ell^2 \ddot{\theta}$

$\ddot{\theta} = -(m\ell)^{-1} F \sin\theta, \quad F = mg$

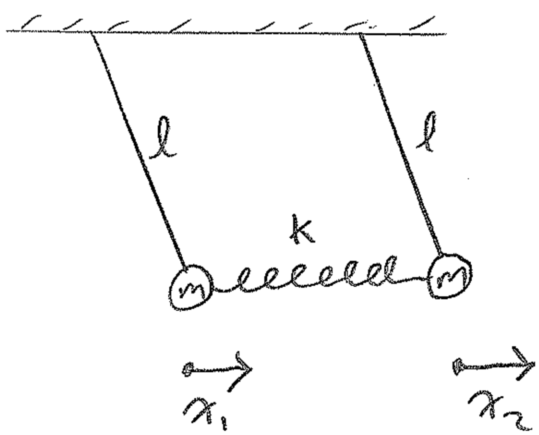
$= -g/\ell \sin\theta$

for $x_1 \ll \ell$, $\sin\theta \simeq x_1$, $\ddot{\theta} \simeq \ddot{x}_1$

$$\Rightarrow \ddot{x}_1 = -g/\ell \, x_1 \xrightarrow{\text{Solve ODE}} x_1(t) = A\sin\omega t$$

$$\omega = \omega_0 = \sqrt{g/\ell}$$

Now consider <u>2 pendulums</u>:



Equations of motion:

i. $m\ddot{x}_1 = \sum \vec{F}_{left}$

ii $m\ddot{x}_2 = \sum \vec{F}_{right}$

Hooke's law

$$\text{i. } m\ddot{x}_1 = -mg/\ell \, x_1 \underbrace{- kx_1 + kx_2}$$

$$= -mg/\ell \, x_1 + k(x_2 - x_1)$$

$$\text{ii. } m\ddot{x}_2 = -mg/\ell \, x_2 + k(x_1 - x_2)$$

Solve <u>system</u> of ODEs: ANSATZ, $x_1(t) = \mathrm{Re}[A_1 e^{i\omega t}]$

$x_2(t) = \mathrm{Re}[A_2 e^{i\omega t}]$, where $\omega = \omega_A$ or ω_B

We define ω_A & ω_B as <u>characteristic</u> or <u>natural frequencies</u>.

<u>Normal mode</u>: the motion, when all degrees of freedom oscillate sinusoidally with the same natural frequency.

We now use the method of determinants to solve,

(1) Put ansatz into coupled ODEs:

$$-m\omega^2 A_1 = -\frac{mg}{\ell} A_1 + k(A_2 - A_1)$$
$$-m\omega^2 A_2 = -\frac{mg}{\ell} A_2 + k(A_1 - A_2)$$

(2) Let the eigenvector be $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and put in matrix form

$$m\omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} k + mg/\ell & -k \\ -k & k + mg/\ell \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

solve for ω (class exercise)

Let $\lambda = m\omega^2$ and solve system by determinants.

$$\det(C - \lambda I) = (k + mg/\ell - \lambda)^2 - k^2$$
$$= \emptyset$$

$$\Rightarrow 2k(mg/\ell - \lambda) + (mg/\ell - \lambda)^2 = 0$$

$$(mg/\ell - \lambda)(2k + mg/\ell - \lambda) = 0$$

$$\lambda = mg/\ell = m\omega_A^2 \quad \text{OR} \quad \lambda = 2k + mg/\ell = m\omega_B^2$$

$$\boxed{\omega_A = \sqrt{g/\ell}} \quad \text{and} \quad \boxed{\omega_B = \sqrt{g/\ell + 2k/m}}$$

$\therefore$ all degrees of freedom must oscillate at ω_A and ω_B

Thus the equations of motion are:

$$A: \quad x_1(t) = A_1 e^{i\omega_A t} \xrightarrow{+} x_2(t) = A_2 e^{i\omega_A t}$$

$$B: \quad x_1(t) = A_1 e^{i\omega_B t} \xrightarrow{+} x_2(t) = A_2 e^{i\omega_B t}$$

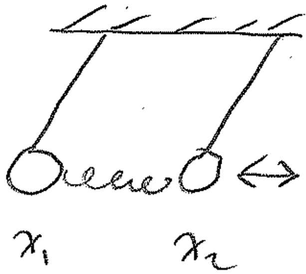
What are the Normal modes, find the eigenvectors

Excercise: case 1, $\lambda = mg/\ell$

$$\Rightarrow 0 = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\Rightarrow 0 = k(A_1 - A_2)$$

$A_1 = A_2$, 1^{st} Normal mode



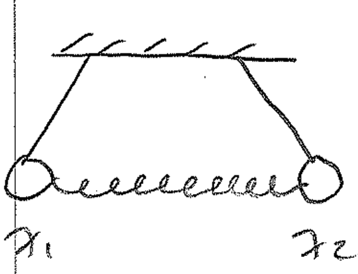
$$x_1 = A \sin \omega_A t$$
$$x_2 = A \sin \omega_A t$$

$\Rightarrow$ spring is not stretched

case 2: $\lambda = 2k + mg/\ell$

$$\Rightarrow 0 = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$0 = -k(A_1 + A_2), \quad A_1 = -A_2$$



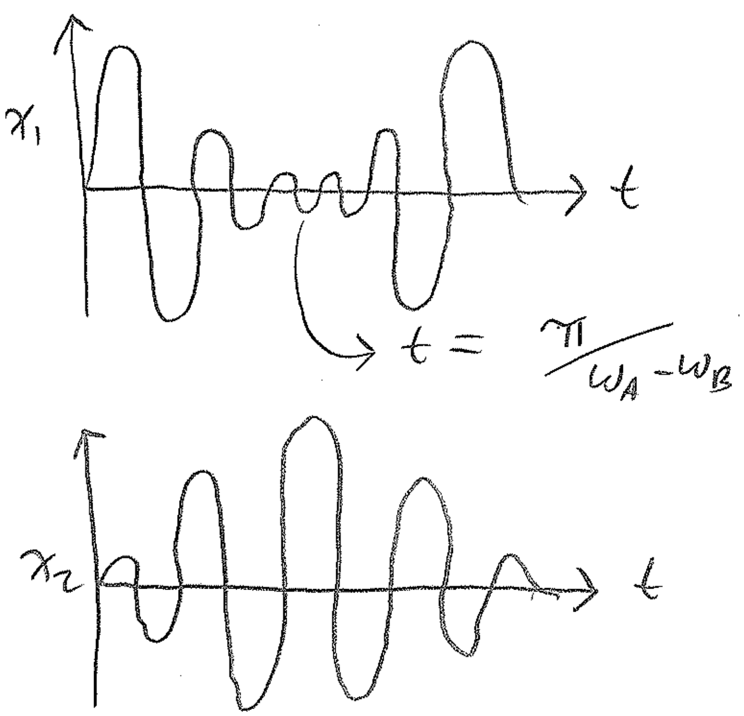
$$x_1 = A \sin \omega_B t$$
$$x_2 = -A \sin \omega_B t$$

Superpositions of normal modes are also solutions

Consider: $x_1 = C \sin \omega_A t + D \sin \omega_B t$

$x_2 = C \sin \omega_A t - D \sin \omega_B t$

lets graph when $C = D$



energy beats between the pendulums.