# Fourier Analysis

Fourier analysis is the decomposition of a general oscillation into harmonic components. In this case, we treat the oscillation as a function of time, so the Fourier decomposition is done in terms of frequencies. A Fourier series is a sum of sinusoidal functions, each of which is a harmonic of the fundamental frequency. A Fourier transform is an integral over a continuous distribution of sinusoidal functions.

## Fourier Series

A Fourier series is appropriate when the system has boundary conditions that limit the allowed frequencies to a discrete set. For a system where the temporal periodicity is $T$ , the Fourier decomposition of a general periodic function is the series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \quad , \tag{1}$$

where the allowed frequencies are harmonics of the fundamental frequency $f_1 = 1/T$

$$\omega_n = n\omega_1 = n2\pi f_1 = n\frac{2\pi}{T} \quad . \tag{2}$$

The expansion coefficients $c_n$ in Eqn. (1) are complex. The real version of the Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n2\pi t}{T}\right) + b_n \sin\left(\frac{n2\pi t}{T}\right) \right] \quad . \tag{3}$$

The expansion coefficients $a_n, b_n, c_n$ are obtained by calculating the overlap integrals (*i.e.*, projections or inner products) of the desired function with the harmonic basis functions

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{n2\pi t}{T}\right)dt \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{n2\pi t}{T}\right)dt \quad . \\ c_n &= \frac{1}{T}\int_0^T f(t)e^{-i\omega_n t}\,dt \end{aligned} \tag{4}$$

## Fourier Transform

A Fourier transform is appropriate when the system has no boundary conditions that limit the allowed wave vectors. In this case, the Fourier decomposition is an integral over a continuum of frequencies:

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}\,d\omega\ , \tag{5}$$

where the expansion function $F(\omega)$ is complex. To obtain the expansion function $F(\omega)$ for a given temporal function $f(t)$ requires the inverse Fourier transform

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega t}f(t)dt\ , \tag{6}$$

which is a projection of the temporal function $f(t)$ onto the harmonic basis functions $e^{i\omega t}/\sqrt{2\pi}$ . The basis functions are orthogonal and normalized in the Dirac sense, which means their projections onto each other are Dirac delta functions

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega' t}e^{-i\omega t}\,dt&=\delta(\omega-\omega')\\ \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t'}e^{-i\omega t}\,d\omega&=\delta(t-t')\end{aligned}\ , \tag{7}$$

whether viewed in the time representation or the frequency representation.

Some typical Fourier transform pairs are shown in Fig. 1 and are listed below (without proper scale factors).

$$\begin{aligned}
\underbrace{f(t)=e^{i\omega_0 t}}_{\text{sinusoid}} \quad &\Leftrightarrow \quad \underbrace{F(\omega)=\delta(\omega-\omega_0)}_{\text{delta function}}\\
\underbrace{f(t)=e^{i\omega_0 t}e^{-\frac{t^2}{2\sigma^2}}}_{\text{Gaussian}} \quad &\Leftrightarrow \quad \underbrace{F(\omega)=e^{-\frac{\sigma^2(\omega-\omega_0)^2}{2}}}_{\text{Gaussian}}\\
\underbrace{f(t)=e^{i\omega_0 t}e^{-\frac{|t|}{\sigma}}}_{\text{exponential}} \quad &\Leftrightarrow \quad \underbrace{F(\omega)=\frac{1}{1+\sigma^2(\omega-\omega_0)^2}}_{\text{Lorentzian}}\\
\underbrace{f(t)=e^{i\omega_0 t};\ |t|<\sigma}_{\text{square pulse}} \quad &\Leftrightarrow \quad \underbrace{F(\omega)=\frac{\sin\{\sigma(\omega-\omega_0)\}}{\sigma(\omega-\omega_0)}}_{\text{sinc}}
\end{aligned}\ . \tag{8}$$

In each case, $F(\omega)$ and $f(t)$ are Fourier transforms of each other following Eqns. (5) and (6). In Fig. 1, only the real part of the function $f(t)$ is plotted and each wave has a central frequency $\omega_0$ .

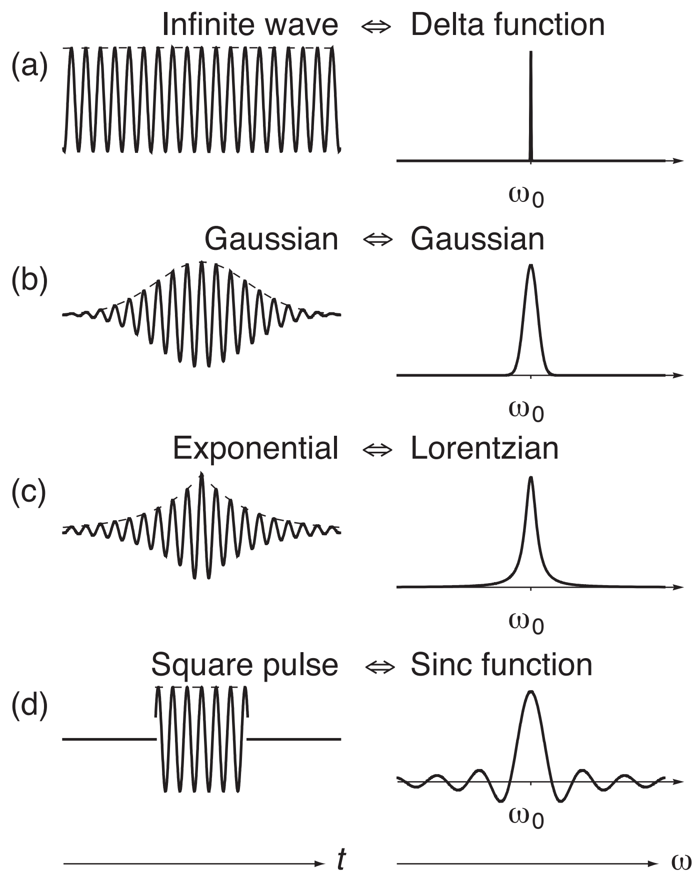



**Figure 1.** Fourier transform pairs: (a) Infinite wave $\leftrightarrow$ delta function, (b) Gaussian $\leftrightarrow$ Gaussian, (c) exponential $\leftrightarrow$ Lorentzian, (d) square pulse $\leftrightarrow$ sinc function.

The temporal extent $\Delta t$ of a function $f(t)$ and the width $\Delta\omega$ of the Fourier transform $F(\omega)$ in frequency space are inversely related through the Fourier uncertainty relation

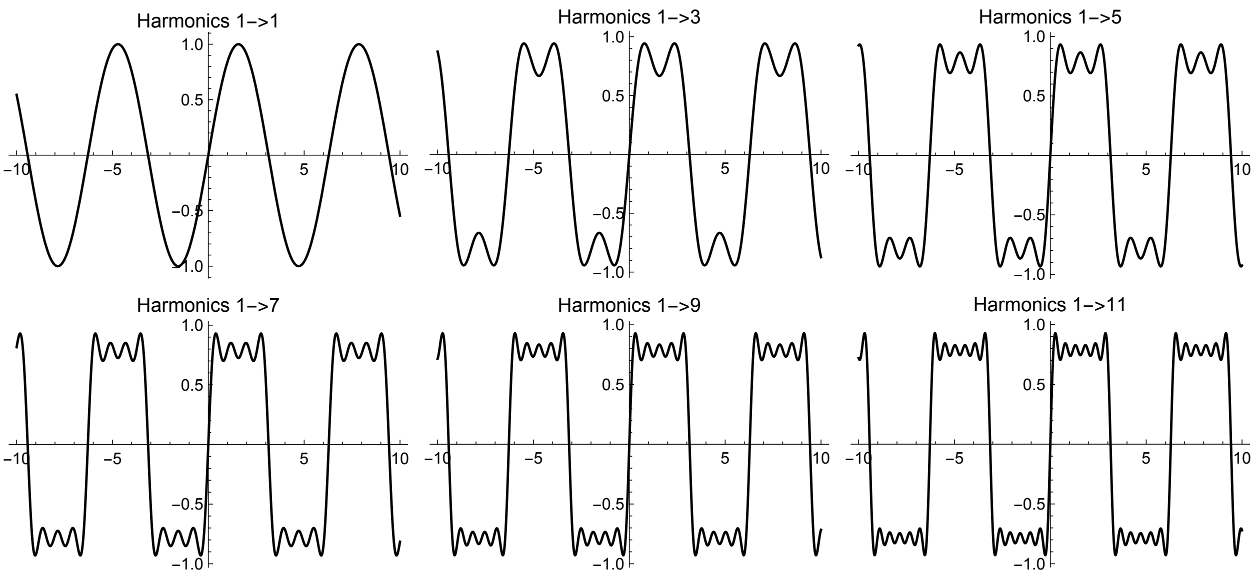
$$\Delta\omega \, \Delta t \geq 1 \, . \tag{9}$$

This relation tells us that if want to make a signal that is confined to a short time, we need to use a wide range of frequencies. In quantum mechanics, this concept is the Heisenberg uncertainty relation.Parseval's theorem says that the power is the same whether calculated in time or frequency space:

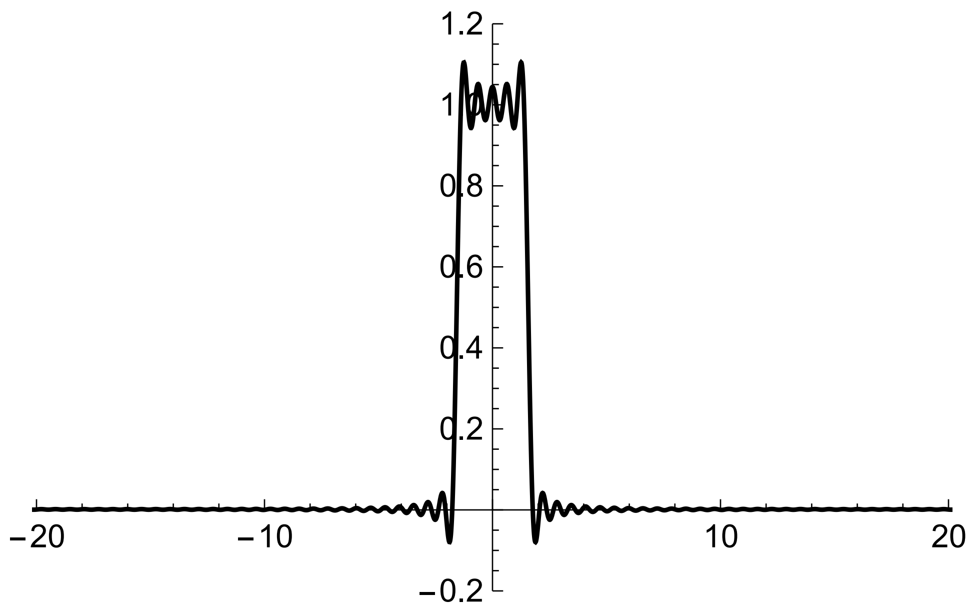
$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega \quad . \tag{10}$$

**Examples**

A square wave can be assembled from a Fourier series of odd harmonics with appropriate coefficients, as shown below:




A single square pulse can be assembled from an infinite assembly of sinusoidal waves:



**Fast Fourier Transform (FFT)**

Fourier analysis on a computer is usually done with Fast Fourier Transform. It is called Fast because it uses a novel algorithm to efficiently calculate the Fourier coefficients. It uses a finite collection of time series data and produces a finite set of Fourier coefficients, so technically it is a Fourier series. It is sometimes called a Digital Fourier Transform.

The input data in the form of a finite time series comprises data taken at equal time intervals $\Delta t$. If the time series includes *N* points, then the total time is $T = N\ \Delta t$. The time interval $\Delta t$ is also known as the sampling time, and the inverse of that time is the sampling frequency $f_{sampling} = 1/\Delta t$.

The FFT uses the *N* time series points to calculate *N* points in the frequency domain. The FFT is complex, so the output has real and imaginary components, or amplitude and phase. We will ignore the phase for this class and focus on the amplitude. So our FFT will have *N*/2 points in a spectrum from zero frequency up to a maximum frequency, which is called the Nyquist frequency. The **Nyquist frequency** is related to the sampling time by the equation

$$f_N = \frac{1}{2\,\Delta t} \tag{11}$$

and hence to the sampling frequency by

$$f_N = \frac{1}{2} f_{sampling} \quad . \tag{12}$$

The spacing between points in the frequency spectrum is $\Delta f = f_N/(N/2)$, which is also $\Delta f = 1/T$ .

The Nyquist frequency is important because it is the largest frequency that you can detect with the FFT method. In other words, the sampling frequency must be at least twice as large as the highest frequency that you expect in your experiment. Unfortunately, if there are frequencies in your experiment larger than the Nyquist frequency, they will still appear in your data, but they will have apparent frequencies different than their actual frequencies. This problem is known as **aliasing** and is similar to the stroboscopic effect that you see at dance parties where the strobe light changes your perception of the dancer's motion.