10

Appendix A

Thevenin's Theorem

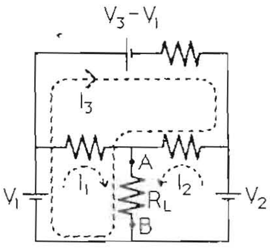


Fig. A1. Loop currents through terminals AB.

Any network can be described by loop currents passing from voltage sources to the terminals AB, hence through a load resistor R_L. An example is shown in figure A.1. In order to establish Thevenin's theorem, it is necessary to show that $V_{AB} = V_{EQ} - R_{EQ} I_{AB}$, where V_{EQ} and R_{EQ} do *not* depend on the load resistance R_L.

Around each current loop, there is a *linear* equation:

$$V_{AB} = V_1 - R_{11}I_1 - R_{12}I_2 - \cdots - R_{1n}I_n$$
$$V_{AB} = V_2 - R_{21}I_1 - R_{22}I_2 - \cdots - R_{2n}I_n$$
$$\vdots$$
$$V_{AB} = V_n - R_{n1}I_1 - R_{n2}I_2 - \cdots - R_{nn}I_n.$$

Because the voltage drops appearing on the right-hand side in the form $R_{ij}I_j$ stop short of the terminals AB, the load resistance R_L does not appear explicitly anywhere in the equations. They can be solved for currents (by simple elimination and substitution), with the results

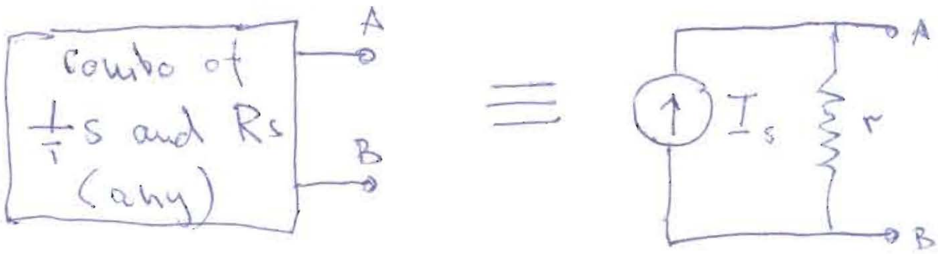
$$I_1 = Y_{11}V_1 + Y_{12}V_2 + \cdots + Y_{1n}V_n - \gamma_1 V_{AB}$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2 + \cdots + Y_{2n}V_n - \gamma_2 V_{AB}$$
$$\vdots$$
$$I_n = Y_{n1}V_1 + Y_{n2}V_2 + \cdots + Y_{nn}V_n - \gamma_n V_{AB}$$

where again none of the coefficients depend on R_L (since R_{ij} did not). Then $I_{AB} = \sum_i I_i$ can be written $B_1V_1 + B_2V_2 + \cdots + B_nV_n - CV_{AB}$. This is of the required form if $C = 1/R_{EQ}$ and $V_{EQ} = (B_1V_1 + B_2V_2 + \cdots + B_nV_n)/C$.

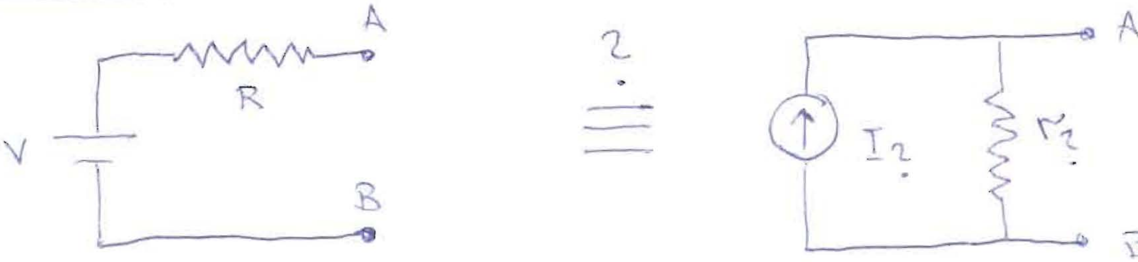
In Chapter 6, it is shown that there is a linear relation between V and I for capacitors and inductors when the notation of complex numbers is used. The linear relation between V and I is the foundation of Thevenin's theorem. Hence the proof carries over to include capacitors and inductors in AC circuits. Fourier's theorem expresses any waveform in terms of AC components, and Thevenin's theorem is therefore generally valid for any network where the relation between V and I is linear. This extends it to small signals in non-linear circuits, following the methods of section 1.11.

Norton's Theorem:

Any combination of batteries & resistances with two terminals ~~…~~ is electrically equivalent to an ideal current source in parallel with a resistance:

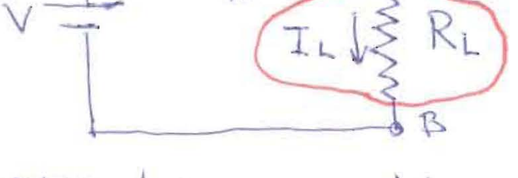


Example:



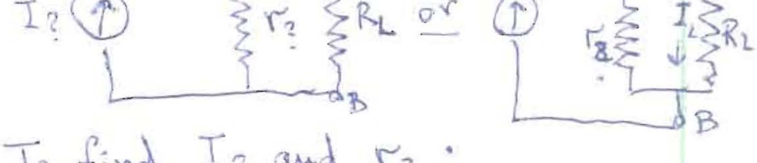
$I_?$, $r_?$ to be found.

Solution: use an arbitrary load resistor R_L:



Obviously: $I_L = \dfrac{V}{R + R_L}$

Use the same load resistor in the "Norton equivalent circuit":

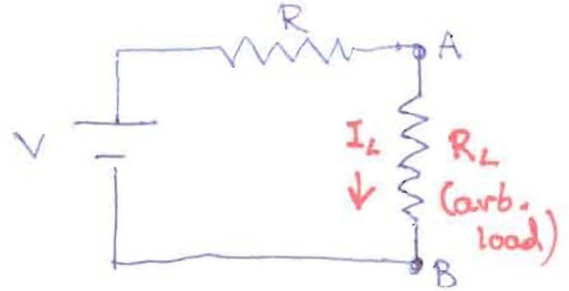


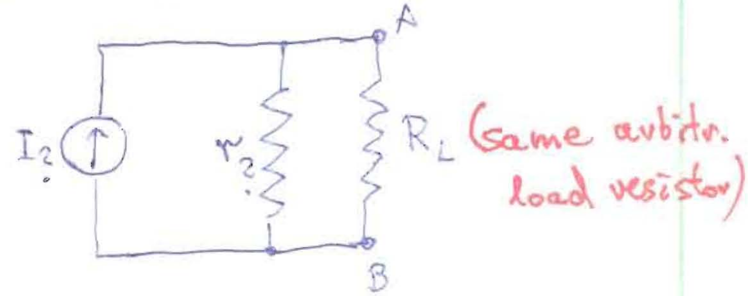
To find $I_?$ and $r_?$:

$$V_{AB} = I_?\,(r_? \parallel R_L) = I_? \frac{r_? \cdot R_L}{r_? + R_L}$$

$$I_L = \frac{V_{AB}}{R_L} = \frac{I_?\,(r_? \cdot R_L)}{R_L\,(r_? + R_L)} = I_? \frac{r_?}{r_? + R_L}$$

continued from the preceding page:





$$I_L = \frac{V}{R+R_L}$$

We get: $I_L = I_z \dfrac{r_z}{r_z + R_L}$

We want to get the same current I_L as in the left...

Note that if we take:

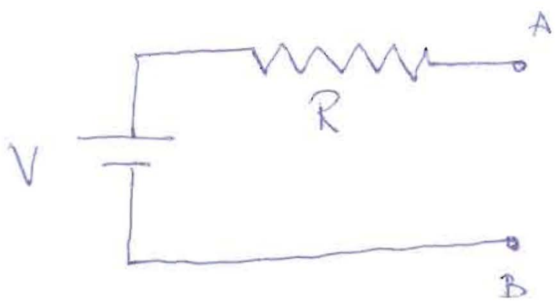
$\boxed{r_z = R}$ and $\boxed{I_z = \dfrac{V}{r_z} = \dfrac{V}{R}}$

We get:

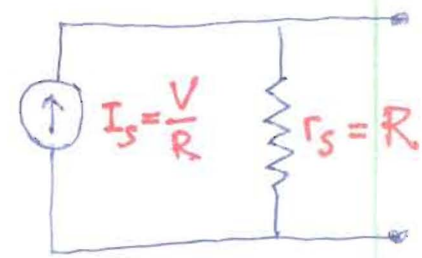
$$I_L = \frac{V}{R} \cdot \frac{R}{R+R_L} = \frac{V}{R+R_L}$$

THE SAME!!!

<u>So, we found:</u>



Norton's $\equiv$ equivalent:



It was a very simple example, but it is a good practice to start with simple things, and gradually go to more complicated ones...

IT'S NOT A "LEARNED GUESS"!

Formal Proof: (of what is used in Page 12)

$$I_L = \frac{V}{R+R_L} \begin{pmatrix}\text{Original}\\ \text{circuit}\end{pmatrix} \text{ and } I_L' = I_2 \frac{r_2}{r_2+R_L} \begin{pmatrix}\text{Norton's}\\ \text{equivalent}\end{pmatrix}$$

Note: V and R have fixed values

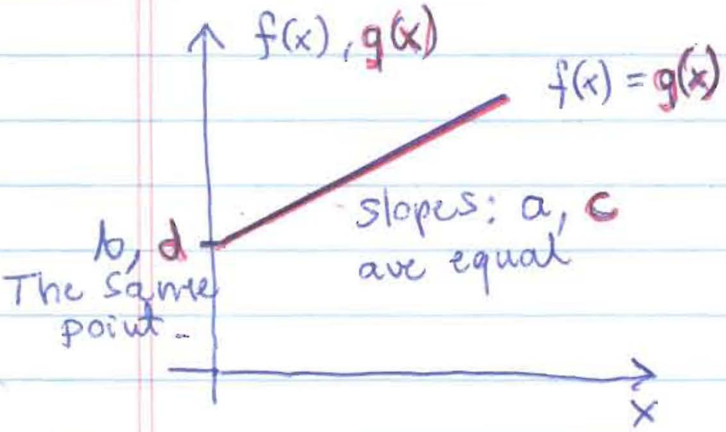
Question: for what I_2 and r_2 values $I_L \equiv I_L'$, i.e., the right-hand sides are always equal, no matter of what the value of R_L is?

If $\dfrac{V}{R+R_L} = I_2 \dfrac{r_2}{r_2+R_L}$, then: $\boxed{\dfrac{R}{V} + \dfrac{1}{V} R_L = \dfrac{1}{I_2} + \dfrac{R_L}{r_2 I_2}}$

(because if $a/b = c/d$, then also $b/a = d/c$).

Note that both sides of the equation are linear functions of R_L

If $f(x) = ax + b$, and $g(x) = cx + d$, then, if $f(x) = g(x)$, it must be: $a = c$ and $b = d$



b & d are intersection points

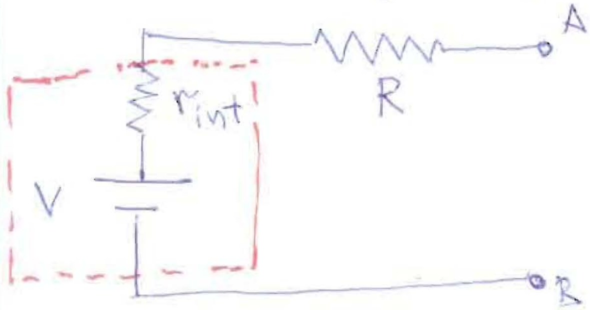
a & c " slopes

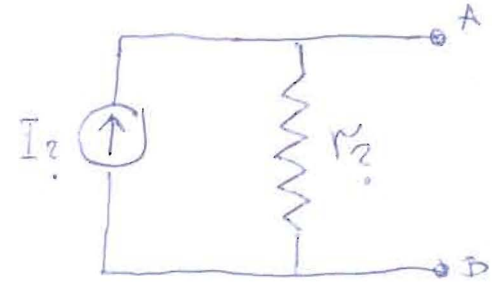
If two linear functions are identical, they intersect the ordinate at the same point, and have the same slopes

Ergo, it must be: $\dfrac{1}{I_2} = \dfrac{R}{V} \Rightarrow I_2 = \dfrac{V}{R}$; and $\dfrac{1}{V} = \dfrac{1}{r_2 I_2} \Rightarrow r_2 = R$

Q.E.D.

A slightly more complicated: the battery from the preceding example has internal resistance:

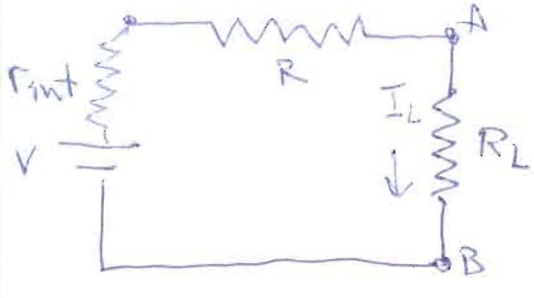


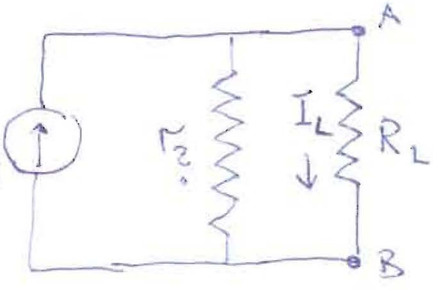


Since the internal resistance is in series with R one can think of r_{int} and R as of a single resistor $r_{int} + R$, and plug it into our previous results to get: $I_? = \dfrac{V}{r_{int}+R}$ and $r_? = r_{int} + R$

But, for training, let us do it "from the beginning", by using an arbitrary load resistor:

For the original circuit:





Essentially same circuit as in the preceding example

Obviously:

$$I_L = \frac{V}{(r_{int} + R) + R_L}$$

Here, the expression for I_L in terms of $I_?$ & $r_?$ is the same as we had before:

$$I_L = I_? \, \frac{r_?}{r_? + R_L}$$

— continued: —

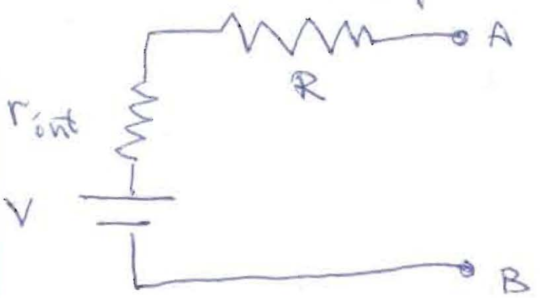
Orig. circuit:

$$I_L = \frac{V}{(r_{int}+R)+R_L}$$

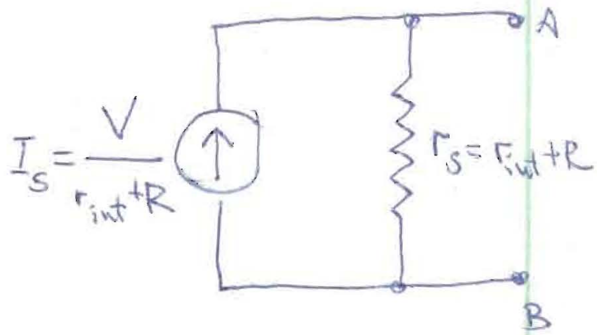
Norton equivalent:

$$I_L = I_? \frac{r_?}{r_? + R_L}$$

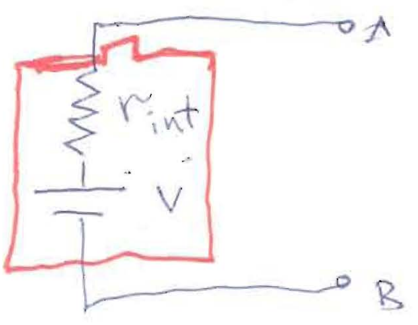
Note that if we take: $r_? = (r_{int}+R)$ & $I_? = \frac{V}{r_{int}+R}$ we get the same expression for I_L in both cases. Therefore:



$\equiv$



Now, what's the "Norton's Equivalent" for a single <u>real</u> battery? Just put $R = 0$:



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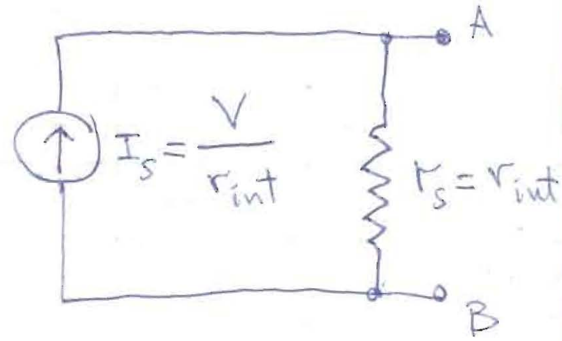
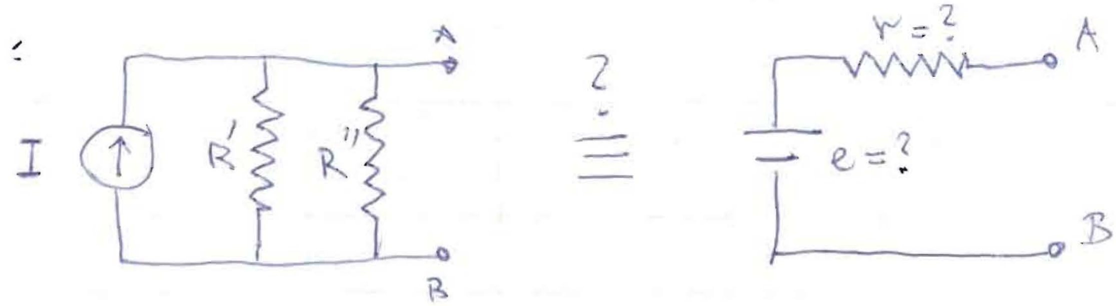
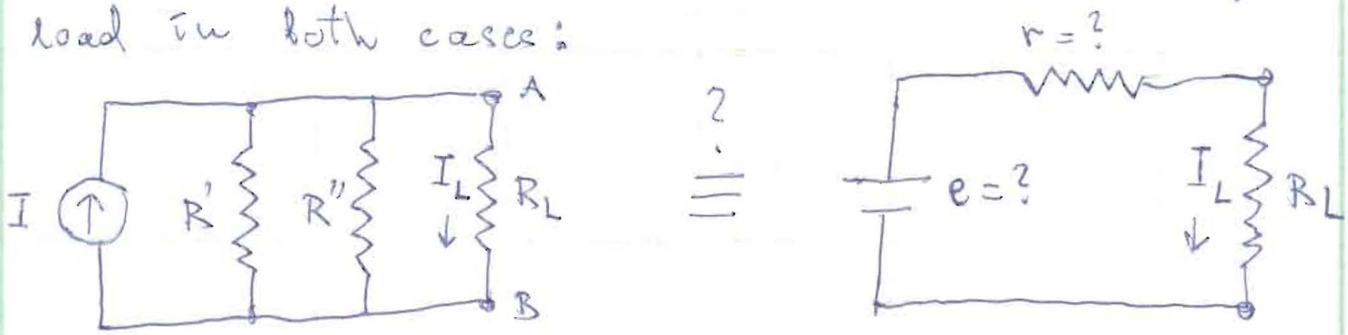


Yet another example – the other way: a Thevenin's equivalent circuit for a one with an ideal current source and two resistors R' and R'' in parallel:



Again, the same "trick" – use the same arbitrary load in both cases:



$$V_{AB} = I \cdot (R' \| R'' \| R_L) \qquad I_L = \frac{e?}{r_? + R_L}$$

$$\frac{1}{(R' \| R'' \| R_L)} = \frac{1}{R'} + \frac{1}{R''} + \frac{1}{R_L}$$

Therefore:

$$V_{AB} = \frac{I}{\frac{1}{R'} + \frac{1}{R''} + \frac{1}{R_L}}$$; The current I_L in load resistor is: $I_L = \frac{V_{AB}}{R_L}$

Combining:

$$I_L = \frac{I}{R_L\left(\frac{1}{R'} + \frac{1}{R''} + \frac{1}{R_L}\right)}$$

Now we need to do quite tedious algebra...

$$I_L = \frac{I}{R_L\left(\frac{R'R_L + R''R_L + R'R''}{R' \cdot R'' \cdot R_L}\right)}$$

$$= I \frac{R' \cdot R'' \cdot \cancel{R_L}}{\cancel{R_L}\left[R_L(R'+R'') + R'R''\right]}$$

$$= \frac{I \cdot R' \cdot R''}{(R'+R'')\left[R_L + \frac{R'R''}{(R'+R'')}\right]}$$

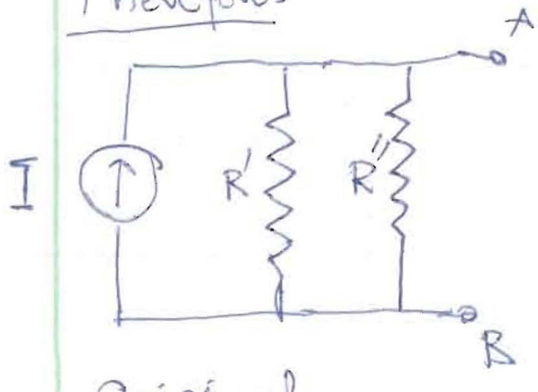
$$= \frac{I \frac{R'R''}{(R'+R'')}}{R_L + \frac{R'R''}{(R'+R'')}}$$

And for the Thevenin equivalent circ., we want: $I_L = \dfrac{e_2}{R_L + r_2}$

So we find that:

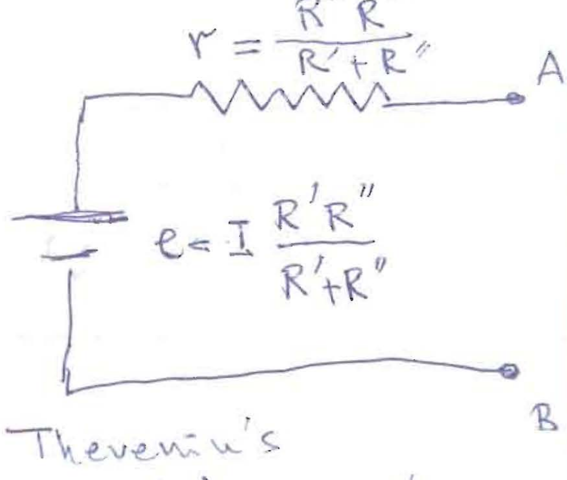
$$e_2 = I \frac{R'R''}{R'+R''} \quad \text{and} \quad r_2 = \frac{R'R''}{R'+R''}$$

Therefore:



Original circuit

$\equiv$



Thevenin's equivalent circuit

Why was all that for? Well, because I want you to look at an illustrative example of how Thevenin's and Northon's theorems may help in solving pretty complicated circuits:

First, on this page and p. 18, coppies from pages from "Circuits, Amplifiers and Gates", a book by D.V. Bugg – once used in the Ph 411 course (and hated by the students, BTW). First, the circuit is solved by using the Kirhoff's Laws:

Another worked example

Figure 1.19(a) shows a rather complicated circuit. As an illustration of all the methods which have been developed in this chapter, we shall find all the currents and the voltages V_A and V_B at nodes by three different methods.

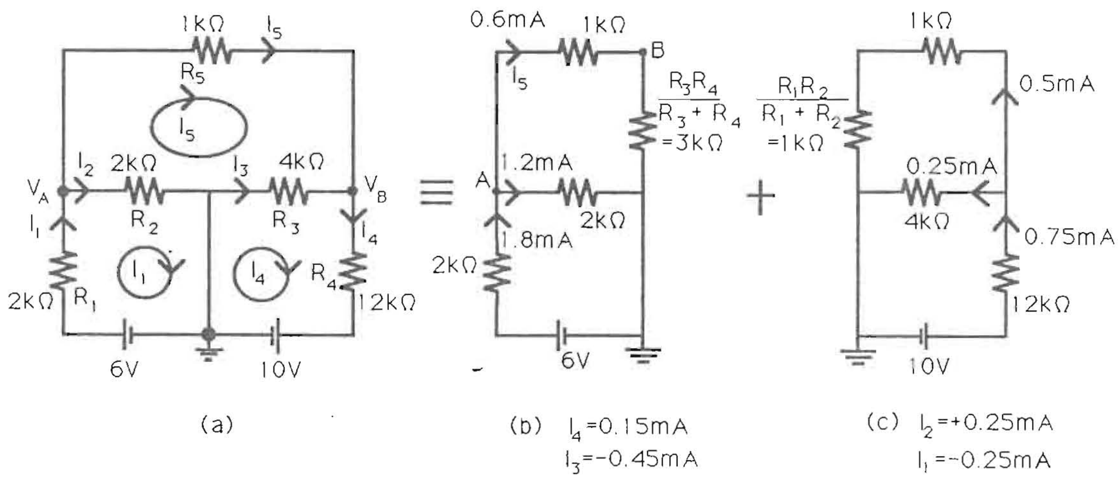


Fig. 1.19 Worked example.

Using superposition, the currents may be obtained from those due to the individual batteries. In (b), the 10 V battery is shorted out and R_3 and R_4 appear in parallel. The top three resistors of (b) provide 2 kΩ in parallel with 4 kΩ between A and earth, i.e. $\frac{4}{3}$ kΩ, so the contribution to $I_1 = 1.8$ mA. It is easy to see how this divides at A; then I_5 splits between 0.15 mA through R_4 and a contribution of -0.45 mA to I_3. In (c), the 6 V battery is shorted, so R_1 and R_2 appear in parallel. The arithmetic of the resulting currents is shown in the figure. The signs of the contributions to I_1 and I_2 are easy to follow from (a) and the sense in which the 10 V battery drives currents. Adding currents from (b) and (c), $I_1 = 1.55$ mA, $I_2 = 1.45$ mA, $I_3 = -0.7$ mA, $I_4 = -0.6$ mA and $I_5 = 0.1$ mA. From these currents, it is simple to find $V_A = 6 - 3.1 = 2.9$ V and $V_B = 10 + I_4R_4 = 2.8$ V.

Suppose instead the problem is to be solved using mesh currents. The ones to choose would be I_1, I_4 and I_5, as shown in (a). Applying Kirchhoff's current law, $I_1 = I_2 + I_5$ and $I_4 = I_3 + I_5$. The values given in the previous paragraph satisfy these relations. Then applying Kirchhoff's voltage law to each loop in turn:

$$\begin{aligned} 6 &= 2I_1 + 2(I_1 - I_5) = 4I_1 - 2I_5 \\ 10 &= -\,12I_4 + 4(I_5 - I_4) = 4I_5 - 16I_4 \\ 0 &= 1I_5 + 4(I_5 - I_4) + 2(I_5 - I_1) = 7I_5 - 4I_4 - 2I_1 \end{aligned}$$

with currents in mA. Solving these three simultaneous equations is tedious. It is however straightforward to substitute the values derived above and demonstrate that the equations are correctly satisfied. Using superposition is really a graphical way of eliminating variables from the simultaneous equations.

The third alternative is to use node voltages V_A and V_B. Then current conservation at these nodes gives

$$\begin{aligned} \frac{6 - V_A}{2} &= \frac{V_A - V_B}{1} + \frac{V_A}{2} \\ \frac{10 - V_B}{12} &= \frac{V_B}{4} + \frac{V_B - V_A}{1}. \end{aligned}$$

The solution of these two simultaneous equations is easy; a check is that the equations are satisfied by the values of V_A and V_B obtained above.

1.10* Non-linear Elements in a Circuit

Superposition is a valuable shortcut, but (as demonstrated below) it only works exactly for circuits containing linear components like resistors, where $V \propto I$. The next chapter develops other powerful shortcuts which again depend on linearity. However, many electronic devices such as diodes and transistors do not obey Ohm's

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(ii) With the batteries replaced by short circuits, R_{EQ} is the resistance across AB, namely 5 kΩ in parallel with 10 kΩ. So $R_{EQ} = 10/3$ kΩ.

(iii) As a check, consider the situation with AB open circuit. In this case, there is a net voltage of 10 V in (a), driving current I in the direction of the arrow through the 5 and 10 kΩ resistors; $I = 10/15$ mA and $V_{AB} = 10 + 5I = 10 + 10/3 = 40/3$ V. This agrees with $I_{EQ}R_{EQ}$ from (i) and (ii).

A circuit can often be simplified quickly and neatly by swopping backwards and forwards between Thevenin and Norton equivalent forms. This is illustrated in figures 2.11(b)–(d). It is a trick worth practising, since it often saves a great deal of algebra. The batteries and resistors of (a) are replaced by equivalent Norton circuits in (b); these are combined in parallel in (c) and then (d) converts back to the Thevenin equivalent form. An important warning is that you must not include the load resistor between terminals A and B in these manipulations: Thevenin's and Norton's theorems apply to the circuits *feeding* terminals AB.

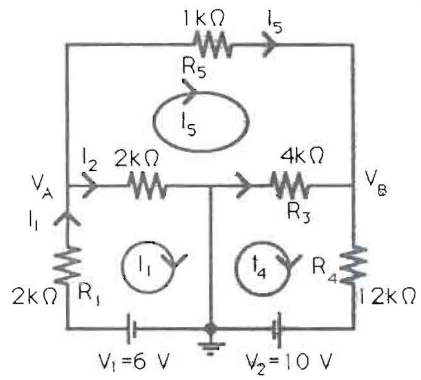

Fig. 2.12. A worked example.

Another worked example

Figure 2.12 reproduces a fairly complicated example from Chapter 1, figure 1.19(a). If all currents and voltages in the circuit are required, it is best to use one of the methods from Chapter 1. Suppose, however, only current I_2 is to be found. It can be obtained straightforwardly by application of Thevenin's and Norton's theorems. The steps are shown in figure 2.13. In (b), V_2 and R_4 are replaced by their Norton equivalent. Then R_3 and R_4 are combined in parallel and (c) returns to the Thevenin equivalent form.

With AB open circuit,

$$
\begin{aligned}
V_{EQ} &= V_1 - \left(V_1 - \frac{V_2R_3}{R_3+R_4}\right) R_1 \left(R_1 + R_5 + \frac{R_3R_4}{R_3+R_4}\right)^{-1} \\
&= 6 - (6 - 2.5)2/(2+1+3) \\
&= \frac{29}{6}\ \mathrm{V}.
\end{aligned}
$$

With the batteries shorted out, R_{EQ} is given by the parallel combination of R_1 with

$$R_5 + R_3R_4/(R_3 + R_4)$$

i.e. 2 kΩ in parallel with 4 kΩ, so $R_{EQ} = (4/3)$ kΩ.

As a check, the current through AB when shorted is I_{EQ}:

$$
\begin{aligned}
I_{EQ} &= \frac{V_1}{R_1} + \frac{V_2R_3}{R_3+R_4}\left(R_5 + \frac{R_3R_4}{R_3+R_4}\right)^{-1} \\
&= 3 + 2.5/4 = 29/8\ \mathrm{mA}.
\end{aligned}
$$

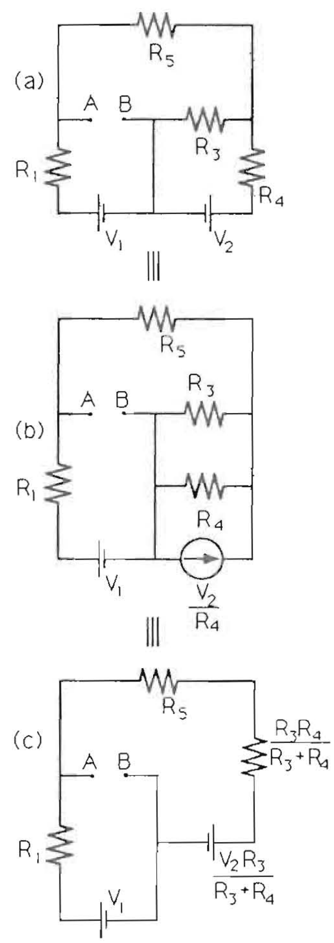

Fig. 2.13. Worked example.

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This agrees with V_{EQ}/R_{EQ} as it should. The arithmetic and algebra are sufficiently tortuous that this is a valuable crosscheck.

Finally, the current I_2 of figure 2.12 is

$$I_2 = V_{EQ}/(R_{EQ} + R_2) = \tfrac{29}{6}\left(\tfrac{4}{3} + 2\right)^{-1} = 1.45 \text{ mA}$$

in agreement with the value obtained in the previous chapter.

Further examples are given in the exercises at the end of the chapter. If you can do question 6, you have mastered the vital points of Chapters 1 and 2 up to here.

2.5 General Remarks on Thevenin's and Norton's Theorems

Extra

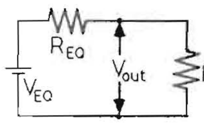


Fig. 2.14. For constant V_{out}, $R_{EQ} \ll R_L$.

(1) Suppose a constant voltage is required across a load R_L, with as little variation as possible when R_L is changed. From figure 2.14, R_{EQ} needs to be small compared with R_L, so that most of V_{EQ} appears across R_L. Thus a constant voltage source should have a low output resistance or **output impedance**, as it is often called.

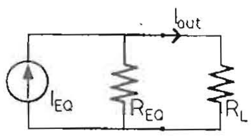
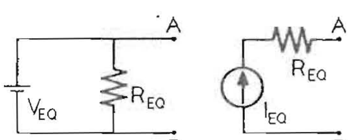


Fig. 2.15. For constant I_{out}, $R_{EQ} \gg R_L$.

(2) Conversely, suppose a constant output current is required, independent of load; this is the case, for example, in supplying a magnet or a motor. From figure 2.15, this demands $R_{EQ} \gg R_L$, or high output resistance.

(3) When a circuit is measured with an oscilloscope or voltmeter, it is desirable to disturb the circuit as little as possible, i.e. draw very little current. This requires the detector to have a high input resistance or **input impedance**. Oscilloscopes and multimeters typically have input resistances of 10^6–10^7 Ω. On the other hand, if an ammeter is inserted into a circuit in order to measure current, we want to disturb the current as little as possible. Therefore an ammeter should have a low resistance.

(4) Although Thevenin's and Norton's circuits are equivalent to any network in the sense of giving the same output voltage and current, they are *not* equivalent as regards power consumption *within* the equivalent circuit. You may easily verify that the power dissipated in the Norton equivalent circuit of figure 2.10(b) is different from that dissipated in the Thevenin equivalent circuit (a). This is because power is non-linear in V or I.



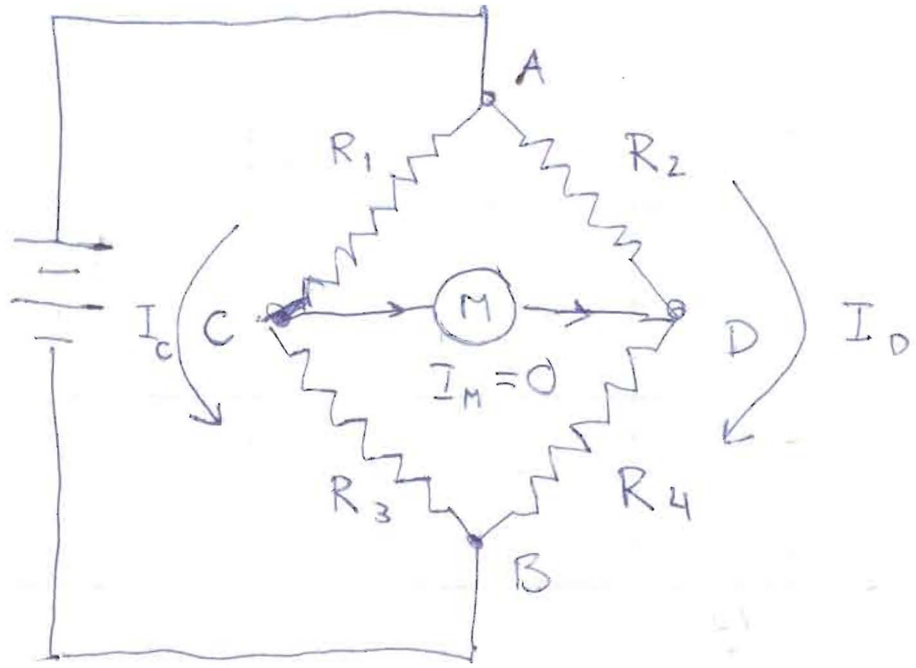
(a) (b)

Fig. 2.16. Wrong choices for equivalent circuits.

(5) Common student howlers are to draw equivalent circuits in the forms shown in figure 2.16. It is worth a moment's thought to see as to why these must be wrong. In the former case, $V_{AB} = V_{EQ}$ independent of load, which gives an absurd result if the terminals are shorted. In the second circuit, $I_{AB} = I_{EQ}$ independent of load, and this is absurd if the terminals are open.

(6) If you encounter a circuit like that in figure 2.16(a) where a resistor is applied directly across a battery, you can ignore the

Wheatstone Bridge – an extremely important circuit in electrical engineering and electronics:



if the bridge is <u>balanced</u>, there is no current flowing through the meter: $I_{CD} = I_M = 0$

So the voltage drop $V_{AC} = V_{AD}$:

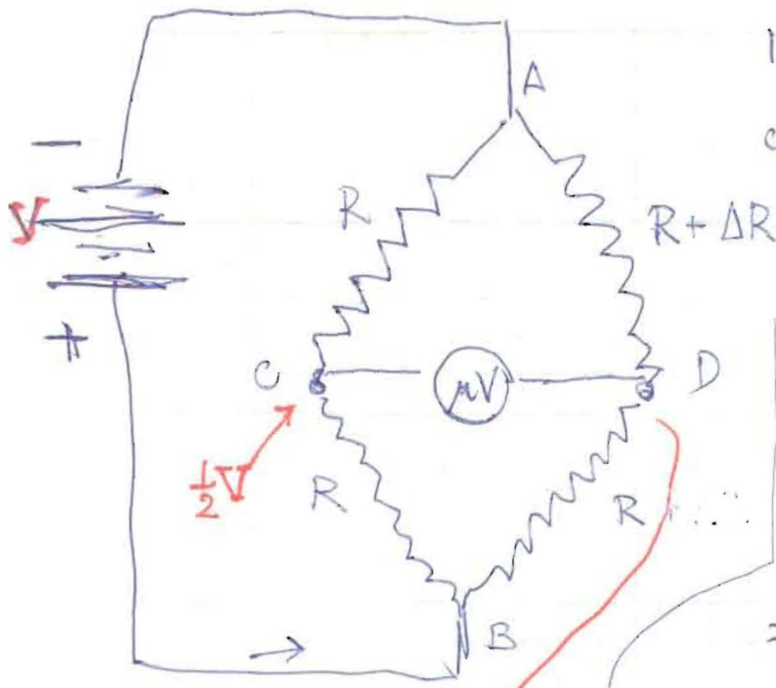
$$R_1 I_C = R_2 I_D$$

and $V_{CB} = V_{DB}$, so that:

$$R_3 I_C = R_4 I_D$$

Solving for I_D / I_C, we get: $\frac{R_1}{R_2} = \frac{R_3}{R_4}$

If R_1, R_2 & R_4 are known, then from the equilibrium condition we get $R_3 = R_4 \frac{R_1}{R_2}$ which can be used for a precise determination of R_3



In fact, a bridge consists of two voltage dividers...

$V_{A} = \frac{1}{2} V$

$V_{AB} = V \cdot \frac{R_{AD}}{R_{DB} + R_{AD}} =$

$= V \frac{R + \Delta R}{(R + \Delta R) + R} =$

$= V \frac{R + \frac{1}{2}\Delta R + \frac{1}{2}\Delta R}{2R + \Delta R} =$

$\frac{1}{2} V + \frac{1}{2} V \frac{\Delta R}{2R + \Delta R}$

if $\Delta R \ll R$:

$\approx \frac{1}{2} V + \frac{1}{4} \frac{\Delta R}{R}$

$\frac{1}{2} V + \frac{1}{4} \frac{\Delta R}{R}$

Differential voltage

$\Delta V = V_{CD} \cong \frac{1}{4} \frac{\Delta R}{R}$

If we have a sensitive meter, e.g. μV-meter or n-V-meter, then we can measure extremely small ΔRs