Relativistic Dynamics:

This is an extremely important part of the special relativity theory.
We will skip the derivations, however, and accept the results without doing the calculations in class (you are encouraged, though, to study Chapter 2.7 by yourself).

The classical (Newtonian) momentum, as we remember, is:

$$\vec{p} = m\vec{v}$$

In the Special Relativity Theory (SRT), the equation takes a new form:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (\text{where } v = |\vec{v}|).$$

It's easy to see that when $v \to 0$ (low speed limit), $\sqrt{1-\frac{v^2}{c^2}} \to 1$

So:

$$\vec{p} \longrightarrow m\vec{v}$$

I.e., the low-speed limit of the relativistic momentum is the <u>classical momentum</u>.

The classical (Newtonian) kinetic energy is:

$$K = \frac{mv^2}{2}$$

The relativistic kinetic energy has a form that is quite different:

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$

At the limit of low speeds, $v \ll c$, this equation should reduce to the classical expression. Does it?

At the first glance: when $v \to 0$, $\sqrt{1-\frac{v^2}{c^2}} \to 1$

so $K \longrightarrow mc^2 - mc^2 = 0$

This is <u>incorrect</u> reasoning. Correct handling of the low-speed limit is formulated in Problem 27 in Chapter 2.

Let's do that. We want to check the behavior of the $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{\frac{1}{2}}$ function for small x values.

Let's use the Taylor's Expansion formula:

$$f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}f''(0)x^2 + \ldots \frac{1}{n!}f^n(0)x^n \ldots$$

$$f(x) = (1-x)^{-\frac{1}{2}} \Rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1) \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\cdot\left(-\frac{3}{2}\right)(1-x)^{-5/2}(-1) \Rightarrow f''(x) = \frac{3}{4}$$

and so on...

So, we can write:

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{4}x^2 + \dots$$

Now, we are interested in the $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ function,

so $x = \frac{v^2}{c^2}$

and we can write:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{4}\frac{v^4}{c^4} + \dots$$

Accordingly:

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 = \cancel{mc^2} + mc^2\frac{1}{2}\frac{v^2}{c^2} + mc^2\frac{3}{4}\frac{v^4}{c^4} + \dots - \cancel{mc^2}$$

$$= \frac{mv^2}{2} + \frac{3}{4}m\frac{v^4}{c^2} + \dots$$

When $v \ll c$, $\frac{v^4}{c^2}$ becomes very small compared with v^2

Let's check! Take $v = 3$ km/s ; pretty high velocity, but much smaller than c ($c = 3 \times 10^5$ km/s).

$$v^2 = 9 \text{ km}^2/\text{s}^2$$

$$\frac{v^4}{c^2} = \frac{81 \text{ km}^4/\text{s}^4}{9 \times 10^{10} \text{ km}^2/\text{s}^2} = 9 \times 10^{-10} \text{ km}^2/\text{s}^2$$

The second term is about <u>one [redacted] billion</u> times smaller than the first one!

So, in the small speed limit only the $\frac{1}{2} m v^2$ term "survives", and the formula indeed agrees with the classical one.

The kinetic energy can be written as:

$$K = E - E_0$$

with $E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$, called the "relativistic <u>total</u> energy" and $$E_0 = mc^2$$ – called the "rest energy".

One <u>EXTREMELY</u> important conclusion from the SRT is (again, we will accept it without proof!) that the <u>total relativistic energy is conserved</u>!

In other words – it sets the equivalence between the mass and energy through the $E_0 = mc^2$ term.

It means that: mass can be converted to other forms of energy (e.g., kinetic) or – energy (e.g., kinetic) can be converted to mass.

Finally, another convenient relation between the total relativistic energy, momentum, and mass, has the form:

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$