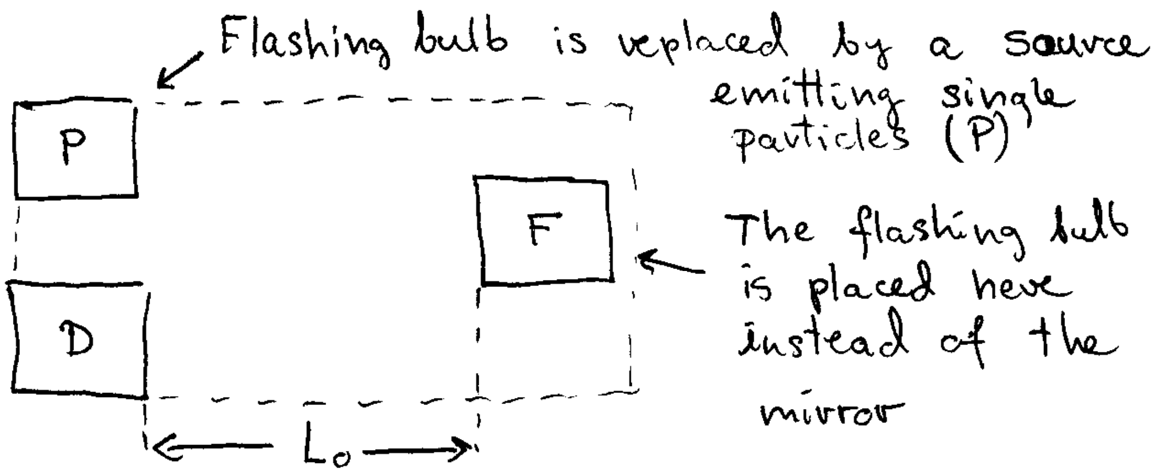
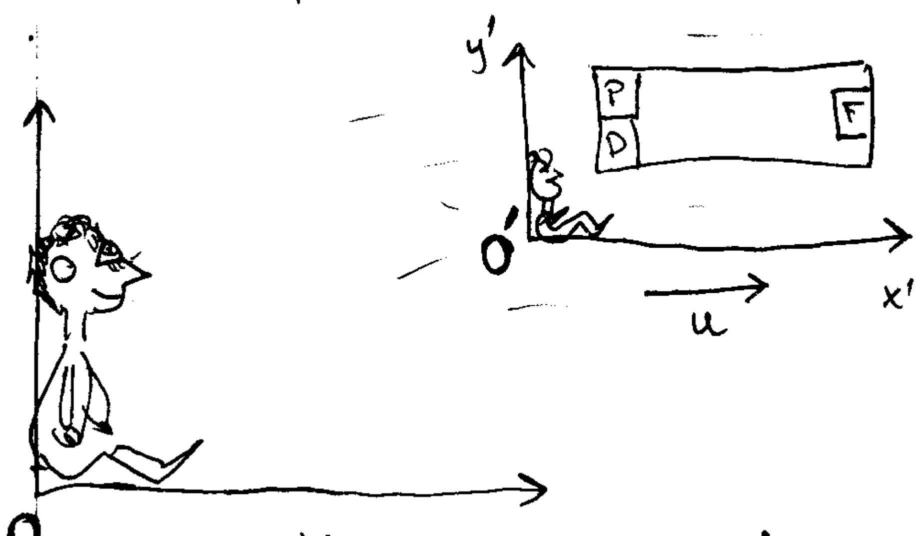
Relativistic Velocity Addition

We now slightly re-design our "light-clock"

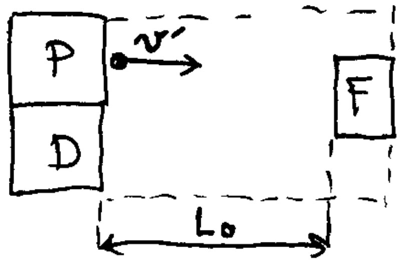


The modified clock is placed in a frame O', moving with velocity u with respect to frame O.

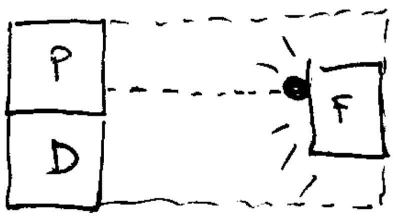


~~The~~ Let's analyze, what sequence of events the observer in O' sees? And what sequence the one in O sees?

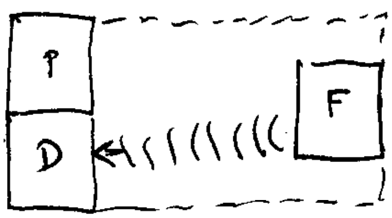
The observer in O' (traveling together with the "clock"):



The source emits a particle. It takes a time $\frac{L_0}{v_1'}$ to reach the flashing bulb



The particle, when reaching the bulb F, triggers a flash



The light signal reaches the detector after L_0/c

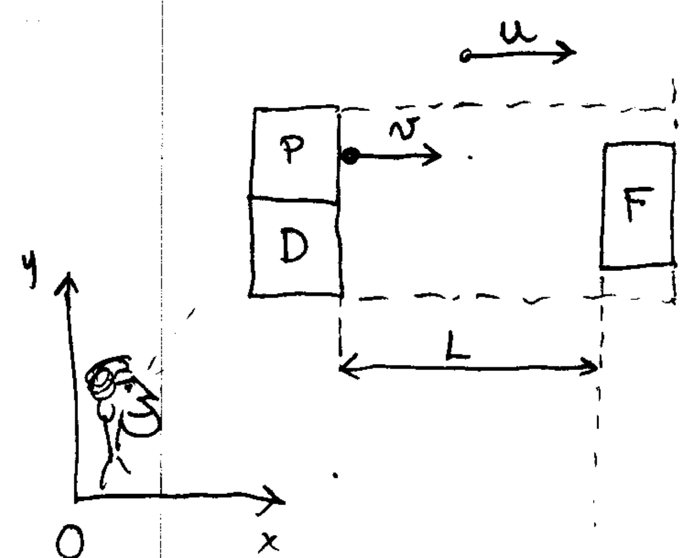
So, the device will produce "ticks" at regular intervals Δt_0, equal:

$$\Delta t_0 = \frac{L_0}{v_1'} + \frac{L_0}{c}$$

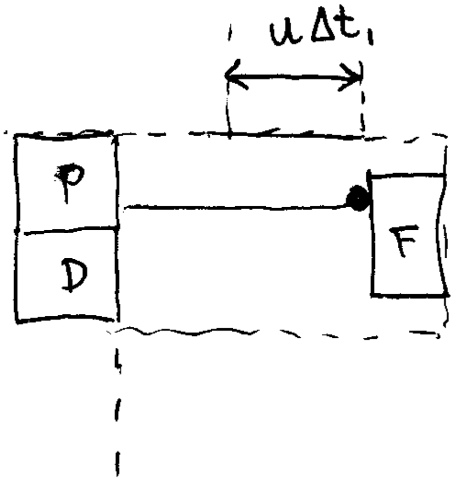
Let's repeat – when the observer moves together with the clock!

Now, how is the sequence seen by the observer in O?

Now, how the observer in the O frame sees it:



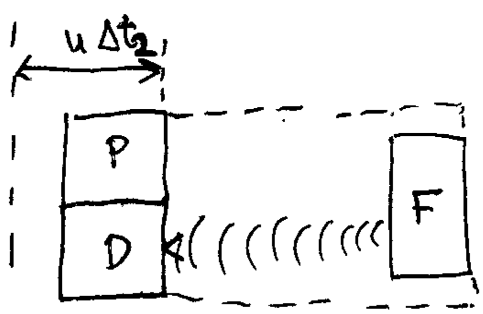
Because the "clock" is moving, the observer sees a different particle velocity (v) than the O' observer (v') and a different clock lengt (due to the length contraction)



The particle reaches the detector after Δt_1 elapses – but the clock has shifted by $u\,\Delta t_1$, so the enite distance traveled is $L + u\Delta t_1$

So, we can write:

$$v \cdot \Delta t_1 = L + u\,\Delta t_1$$



The particle triggers a flash, which reaches the detector after a Δt_2 time passes. But the detector shifts by then by $u \cdot \Delta t_2$

So, the total distance traveled by the light pulse, for the observer O, is $L - u\Delta t_2$

We can write then:

$$c\,\Delta t_2 = L - u\,\Delta t_2$$

We have two equations:

$$v\,\Delta t_1 = L + u\,\Delta t_1$$
$$c\,\Delta t_2 = L - u\,\Delta t_2$$

Let's solve them for Δt_1 and Δt_2:

$$\Delta t_1 = \frac{L}{v - u}$$

$$\Delta t_2 = \frac{L}{c + u}$$

The interval between the tics is

$$\Delta t = \Delta t_1 + \Delta t_2$$

So:

$$\Delta t = \frac{L}{v-u} + \frac{L}{c+u}$$

$$= L\,\frac{c+u+v-u}{(v-u)(c+u)} = L\,\frac{c+v}{(v-u)(c+u)}$$

So, we have now two equations for the "tick period", in the O' frame and O frame, respectively:

$$\Delta t_0 = \frac{L_0}{v'} + \frac{L_0}{c} = L_0 \left(\frac{v' + c}{v' c} \right) \qquad (*)$$

and

$$\Delta t = L \frac{c + v}{(v - u)(c + u)} \qquad (**)$$

In addition, we have the equations for time dilation, and for length contraction, respectively:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad ({*}^{*}{*})$$

$$L = L_0 \sqrt{1 - \frac{u^2}{c^2}} \qquad \left(\begin{matrix} ** \\ ** \end{matrix}\right)$$

From these four, we want now to obtain the relation between v, v' and u

In the book, it is written "after doing the algebra"...

"The algebra" appears to be straightforward, but rather time-consuming...

First, combine Eqs. (**) and (***):

$$\frac{\Delta t_0}{\sqrt{1-\frac{u^2}{c^2}}} = L \frac{c+v}{(v-u)(c+u)}$$

Now, insert Eq. (*) to the left side, and Eq. (****) to the right side:

$$\frac{\cancel{L_0}\left(\frac{v'+c}{v'c}\right)}{\sqrt{1-\frac{u^2}{c^2}}} = \cancel{L_0}\sqrt{1-\frac{u^2}{c^2}} \frac{c+v}{(v-u)(c+u)}$$

L_0 cancels out; multiply both sides by $\sqrt{1-\frac{u^2}{c^2}}$:

$$\frac{v'+c}{v'c} = \left(1-\frac{u^2}{c^2}\right) \frac{c+v}{(v-u)(c+u)}$$

$$\boxed{\text{Note that: } 1-\frac{u^2}{c^2} = \frac{c^2-u^2}{c^2} = \frac{(c-u)(c+u)}{c^2}}$$

So:

$$\frac{v'+c}{v'c} = \frac{(c-u)\cancel{(c+u)}(c+v)}{c^2(v-u)\cancel{(c+u)}}$$

$$\frac{v'+c}{v'} = \frac{(c-u)(c+v)}{c(v-u)}$$

Multiply both sides by $v'c\,(v-u)$:

$$c\,(v-u)(v'+c) = v'\,(c-u)(c+v)$$

Carry out the multiplications:

$$c\,(v v' + v c - u v' - u c) = v'\,(c^2 + c v - u c - u v)$$

$$\cancel{c v v'} + v c^2 - \cancel{c u v'} - u c^2 = v' c^2 + \cancel{c v v'} - \cancel{u c v'} - u v v'$$

$$c^2 v - c^2 u = c^2 v' - u v v'$$

Re-group:

$$c^2 v + u v v' = c^2 v' + c^2 u$$

$$c^2 v\left(1 + \frac{u v'}{c^2}\right) = c^2 (v' + u)$$

So:

$$\boxed{v = \frac{v' + u}{1 + \frac{u v'}{c^2}}}$$

Hurray!!!
The same as
Eq. (2.17) in the
book!