

The Heisenberg Uncertainty Relationship

You have certainly heard of this important relationship before - it states that: ~~the~~ $\Delta x \cdot \Delta p_x \geq \hbar$ (where $\hbar \equiv \frac{h}{2\pi}$).

Here, Δx is the "uncertainty of the particle position" - in other words, the precision with which the particle position can be determined;

and Δp_x is the "uncertainty of the particle momentum" - or, the precision with which the momentum component in the x direction can be measured.

The Heisenberg Uncertainty Relationship states that the position x and momentum p_x cannot be both determined with an unrestrictedly good precision. Even the best possible apparatus would not help here: there is always a tradeoff!

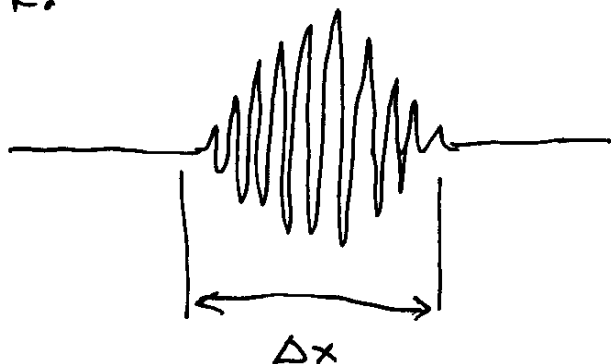
High precision in position determination means that the momentum cannot be precisely determined - and vice versa.

The two uncertainties are such that their product will always be $\Delta x \cdot \Delta p_x \geq \hbar$.

And this is not because our apparatus is not perfect. This is a Law of Nature!

A proof of Heisenberg Uncertainty Principle can be done in several different ways. The most common way is by considering wave packets. We will not discuss the details - I want only to outline the reasoning scheme used in this case.

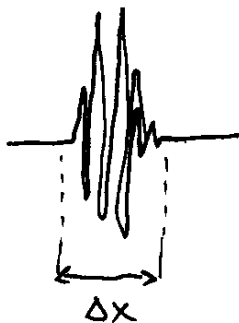
Let's consider a wave packet obtained by superposing de Broglie waves from a certain range of wavevectors Δk :



This wave packet is "spread" within a certain range of x coordinates, Δx . The range Δx

can be thought of as the measure of the position uncertainty. The particle is somewhere within this range.

But, say, we think that Δx is ~~too wide~~ "too wide". We would like to know the particle position with a better accuracy. It would happen if the wave packet were narrower:



But a narrower wavepacket corresponds to a wider range ^(Δk) of wavevectors of constituent waves!

It turns out that for wave packets, the product of Δx and Δk can never be smaller than the minimum value:

$$\Delta x \cdot \Delta k \geq 1$$

For de Broglie waves, $p = \hbar k$, so $\Delta k = \frac{\Delta p}{\hbar}$

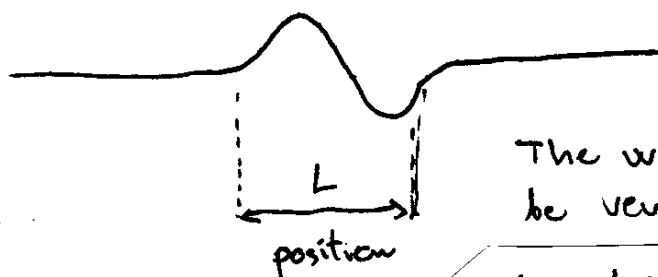
So:

$$\Delta x \cdot \frac{\Delta p}{\hbar} \geq 1 \Rightarrow \Delta x \cdot \Delta p \geq \hbar$$

To translate this into "human language":

Let's consider a "classical analogy" - wavepackets on water surface:

- 1) A "wave packet" consisting only of a single crest and a single trough:

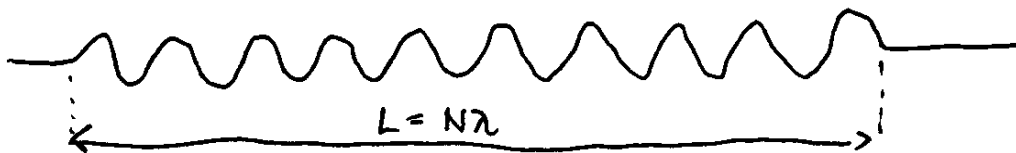


The wavelength can not be very accurately determined

$$\Delta x \sim L \sim \lambda; \Delta \lambda \sim \Delta L$$

$$\text{so: } \Delta x \cdot \Delta \lambda \sim \lambda \cdot \Delta L$$

- 2) A "wave packet" consisting of N waves:



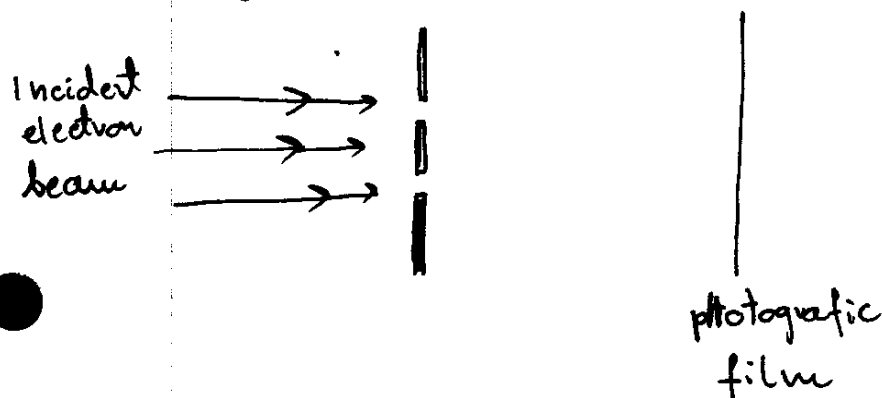
$$\text{Here } \Delta x \sim L \approx N\lambda \text{ and } \Delta \lambda \sim \frac{\Delta L}{N}$$

$$\text{So: } \Delta x \cdot \Delta \lambda \sim N\lambda \cdot \frac{\Delta L}{N} = \lambda \Delta L \text{ same as before.}$$

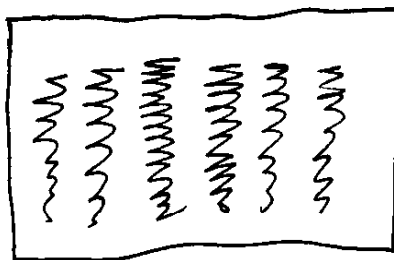
What is the physical interpretation of the de Broglie wave?

This question was the focus of a long-lasting dispute

Consider the double-slit experiment with electrons:



After running the experiment for a long time, and developing the film, we observe "fringes" on it:



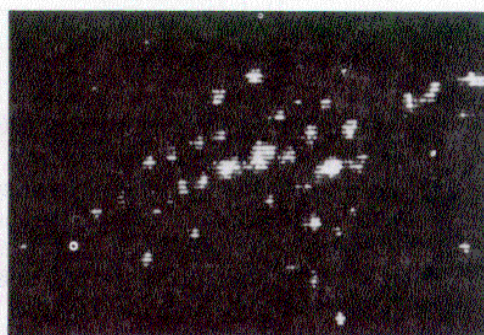
The spacing between the fringes agrees with the de Broglie wavelength of the electrons used. But what does the film show? Does it show "actual de Broglie waves"?

Well, if we conduct the experiment for a short time, we observe only a few "dots" on the film.

Only after a longer time would the "dots" merge and form the "fringes".



(a)



(b)



(c)

Figure 2.8 The genesis of an electron interference pattern in the double-slit experiment. Each figure shows the pattern formed at the detector at successive times. Note the seeming randomness of the pattern at short time intervals. [From P. G. Merli, G. F. Missiroli, and G. Pozzi, *Amer. Jour. Phys.*, **44**, 306 (1976). Used with permission.]

Well - only after a long dispute physicists found an answer, that was finally accepted by the broad community of scientists: namely, the value of the de Broie wavefunction $\psi(x)$ at a given point x expresses the probability of finding the particle at this point:

$$\left(\begin{array}{l} \text{probability of finding} \\ \text{the particle at } x \end{array} \right) = |\psi(x)|^2$$