MTH 654

Numerical Methods for Inverse Problems

HW 2

The objective of this project is to help students familiarize themselves with the theoretical and computational concepts of optimization. In addition, students will gain experience using standard implementations of basic algorithms.

1. Apply Newton's method to find a minimum of $f(x) = \sin^2(x)$ using
   (a) analytic first and second derivatives
   (b) analytic first and forward difference second derivatives
   (c) forward difference first and second derivatives

   Use an initial guess of your choice. For each case above, graph the iteration history of $f(x)$ using finite difference step sizes of $h = 10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}$ on the same plot. Repeat for $f'(x)$. It may also help to make a table similar to:

h	$D(f)$	$D_2(f)$	x	f(x)	fp(x)	fpp(x)
0	0	0				
0.1	0	1				
0.01	0	1				
0.0001	0	1				
1e-08	0	1				
0.1	1	1				
0.01	1	1				
0.0001	1	1				
1e-08	1	1				

   Comment on what you observe.

2. Repeat the previous problem with $f(x) = \sin^2(x) + 10^{-4} rand$ where $rand$ denotes a random number (e.g, `randn(1)`).

3. Show that if $b \in \mathbb{R}^N$ and the $N \times N$ matrix A is symmetric and has a negative eigenvalue, then the quadratic functional

$$m(x) = x^T A x + x^T b$$

   does not have a minimizer. Show that if u is an eigenvector corresponding to a negative eigenvalue of the Hessian, then u is a direction of negative curvature.

4. Suppose $\epsilon(x_c)$ and $\Delta(x_c)$ are errors in the computation of the gradient and the Hessian, respectively, at the Newton iterate x_c. Let the Standard Assumptions hold. Prove that there are constants K,$\delta > 0$ such that if $x_c \in B_\delta(x_c)$ and

$$\|\Delta(x_c)\| < \|(\nabla^2 f(x^*))^{-1}\|^{-1}/4 \tag{1}$$

then

$$\|e_+\| \leq K \left(\|e_c\|^2 + \|\Delta(x_c)\|\|e_c\| + \|\epsilon(x_c)\|\right). \tag{2}$$