MTH 656

Numerical Methods for Inverse Problems

Homework 1

Due: Apr 19

1. Do [V] 1.12: Confirm that the operator representation

$$\mathbf{f}_\alpha = (K^T K + \alpha I)^{-1} K^T \mathbf{d} \tag{1.15}$$

is equivalent to the Tikhonov filter representation

$$\begin{aligned}\mathbf{f}_\alpha &= V \operatorname{diag}(w_\alpha(s_i^2) s_i^{-1}) U^T \mathbf{d} \\ &= \sum_{i=1}^{n} w_\alpha(s_i^2) s_i^{-1} (\mathbf{u}_i^T \mathbf{d}) \mathbf{v}_i \end{aligned} \tag{1.10}$$

$$w_\alpha(s_i) = \frac{s^2}{s^2 + \alpha}. \tag{1.13}$$

To do this, use properties of the SVD to verify that

$$(K^T K + \alpha I)^{-1} K^T \mathbf{d} = V \operatorname{diag}(s_i / (s_i^2 + \alpha)) U^T \mathbf{d}.$$

2. Confirm that the normal equations

$$A^T A \mathbf{f} = A^T \mathbf{d}$$

are equivalent to the least squares problem of minimizing

$$D(\mathbf{f}) = \|\mathbf{d} - A\mathbf{f}\|_2^2$$

by showing that

$$\frac{\partial D}{\partial f_i} = 0$$

can be written

$$\sum_{\ell=1}^{n} \left(\sum_{k=1}^{m} a_{ki} a_{k\ell} \right) f_\ell = \sum_{k=1}^{m} a_{ki} d_k.$$

3. Confirm (possibly using the result above) that minimizing the Tikhonov functional

$$T_\alpha(\mathbf{f}) = \|\mathbf{d} - A\mathbf{f}\|_2^2 + \alpha\|\mathbf{f}\|_2^2 = \left\| \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} \mathbf{f} - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2^2$$

is equivalent to solving the so-called *regularized normal equations*

$$\left(A^T A + \alpha I\right) \mathbf{f} = A^T \mathbf{d}.$$

4. Filtering can be described as replacing Σ^+ with Σ_α^+ in $\mathbf{f} = V\Sigma^+ U^T \mathbf{d}$, where

$$\Sigma^+ = \operatorname{diag}(\sigma_k^+)$$

with

$$\sigma_k^+ = \begin{cases} \frac{1}{\sigma_k} & \text{for } \sigma_k \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Give an expression for Σ_α^+ in the case of *Truncated SVD*, i.e., when $w_\alpha(s^2) = H_\alpha(s^2)$.
   (b) Give an expression for Σ_α^+ in the case of $w_\alpha(s^2) = \min\{1, \frac{s^2}{\alpha}\}$.

5. Sketch the two filter functions described in 4 on a *semilogx* plot (similar to Figure 1.3). Which approach leads to a more computationally intensive solution method?

6. Verify that the two filter functions described in 4 each satisfy equation

$$w_\alpha(s^2) s^{-1} \leq \alpha^{-1/2}, \tag{1.21}$$

and therefore the method, with

$$\alpha = \delta^p, \quad 0 < p < 2 \tag{1.23}$$

is convergent. (Note that bound for the latter filter function is tight.)

7. Verify that the filter function described in 4b defines a regularization method that satisfies

$$\|\mathbf{e}_\alpha^{\text{trunc}}\|_2^2 \leq \frac{4\alpha}{27}\|\mathbf{z}\|_2^2,$$

where $\mathbf{z}$ is defined in

$$\mathbf{f}_{\text{true}} = K^T \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^n. \tag{1.25}$$

Hint: see (1.26) and Exercise 1.7. This is a more restrictive bound than that for TSVD.