MTH 656

Numerical Methods for Stochastic and Random Differential Equations

Homework 1

Due: May 9

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1. Download `KLCodes.zip` from the course website. Given a set of predictions of stream inflows (resulting, say, from equally probable weather forecast realizations), represent the uncertainty in the random field in terms of a finite number of continuous random variables via a Karhunen-Loeve expansion. This expansion can be used to propagate uncertainty through a system, but can also be sampled to produce a large ensemble of realizations with statistics conforming to the original data (data extrapolation). See `KLdriver.m`.

   (a) How many terms are needed to retain 70% of the variance, as measured in the Frobenius norm? How many for 99%?

   (b) Modify the code to display realizations sampled from the truncated KL expansion which retains only 70% of the variance. Comment qualitatively on these realizations compared to those sampled from the full 5-term expansion. (In particular, why is something undesirable happening? Does including one more term fix the problem?)

   (c) Change the correlation length used to generate the simulated data (indirectly by changing the third `ConfidenceInterval` in `GenerateEnsembles.m`) via multiplication by 0.5 and 2. Which corresponds to the smaller correlation length? Comment on the number of terms *necessary to retain* in the resulting KL expansion for each case.

   (d) In practice it is necessary to preserve positivity of inflows (there is a non-zero probability of negative inflows when using Gaussian random variables in the KL expansion). This can be accomplished by using the logarithm of the data in the construction of the KL expansion. Modify `KLdriver.m` by replacing `ProdData` with its `log`. (Be sure to modify the plots of the sampled realizations at the very end in order to be able to compare these results with the original code.) Comment on the number of terms necessary to retain in this KL expansion and the quality of the sampled realizations (you may have to look carefully at the tails and/or increase `ConfidenceLevel` in `GenerateEnsembles.m` and/or change `rng('default')` to `rng('shuffle')` in `KLdriver.m` to see a difference).

2. Consider the linearized predator-prey model

$$\begin{aligned} \dot{\vec{w}} + A\vec{w} &= 0, \qquad 0 < t < T \\ \vec{w}(0) &= \vec{w}_0 \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}.$$

Here $b > 0$ and $c > 0$ are parameters measured in the field and subject to a relative error with noise level r and bias m, i.e., $\hat{b} = (r\xi + m)b$ and $\hat{c} = (r\xi + m)c$, with $\xi \sim U[-1, 1]$. The random ODE incorporating uncertainty is

$$\dot{\vec{w}} + (r\xi + m)A\vec{w} = 0.$$

(a) Show by example that the modal equations are

$$\dot{\vec{w}}_N + A \otimes (rM_N + mI_N)\vec{w}_N = \vec{0}$$

by expanding each term of $\vec{w} = [u, v]^T$ with a Polynomial Chaos expansion of degree two and substituting into the RODE.

(b) Use `ode45` or similar to simulate the modal equations and plot the expected values with the following parameter values: $u_0 = v_0 = 100$, $T = 250$, $b = c = 0.5$, and the following cases

i. $r = 0$, $m = 1$

ii. $r = 0.01$, $m = 1$

iii. $r = 0$, $m = 1.1$

iv. $r = 0.01$, $m = 1.1$

For each case, comment on qualitative behavior of solutions, in particular, are solutions stable oscillations, or decaying? Does this agree with the theory that says if $r < m$ then the sign of the real part of the eigenvalues is preserved?