# CALCULATION OF FUNCTIONS

Using hand calculations, a hand calculator, or a computer, what are the basic operations of which we are capable? In essence, they are addition, subtraction, multiplication, and division (and even this will usually require a truncation of the quotient at some point). In addition, we can make logical decisions, such as deciding which of the following are true for two real numbers $a$ and $b$:

$$a > b, \quad a = b, \quad a < b$$

Furthermore, we can carry out only a finite number of such operations. If we limit ourselves to just addition, subtraction, and multiplication, then in evaluating functions $f(x)$ we are limited to the evaluation of polynomials:

$$p(x) = a_0 + a_1 x + \cdots a_n x^n$$

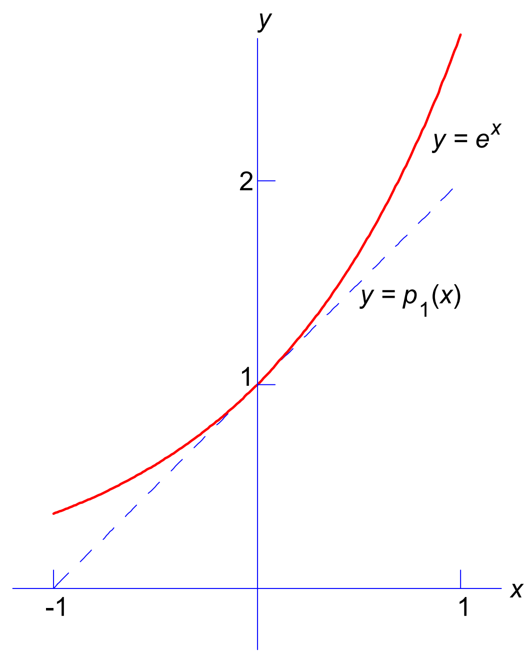
In this, $n$ is the degree (provided $a_n \neq 0$) and $\{a_0, ..., a_n\}$ are the coefficients of the polynomial. Later we will discuss the efficient evaluation of polynomials; but for now, we ask how we are to evaluate other functions such as $e^x$, $\cos x$, $\log x$, and others.

# TAYLOR POLYNOMIAL APPROXIMATIONS

We begin with an example, that of $f(x) = e^x$ from the text. Consider evaluating it for $x$ near to 0. We look for a polynomial $p(x)$ whose values will be the same as those of $e^x$ to within acceptable accuracy.

Begin with a linear polynomial $p(x) = a_0 + a_1 x$. Then to make its graph look like that of $e^x$, we ask that the graph of $y = p(x)$ be tangent to that of $y = e^x$ at $x = 0$. Doing so leads to the formula

$$p(x) = 1 + x$$



y
y = e^x
2
y = p_1(x)
1
-1
1
x

Continue in this manner looking next for a quadratic polynomial

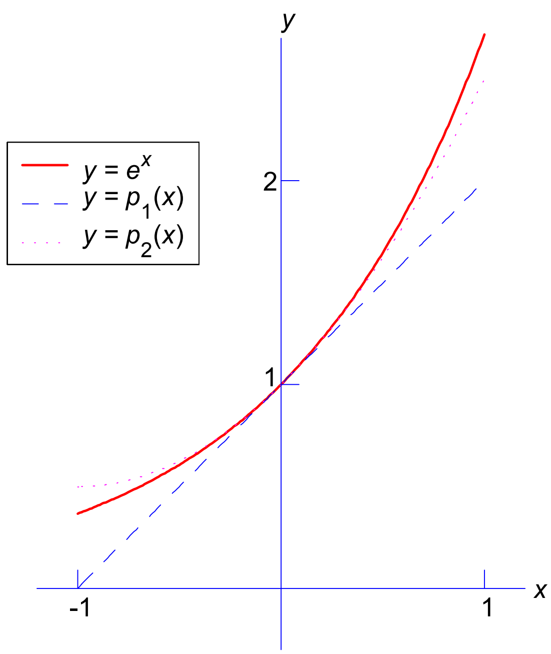
$$p(x) = a_0 + a_1 x + a_2 x^2$$

We again make it tangent; and to determine $a_2$, we also ask that $p(x)$ and $e^x$ have the same "curvature" at the origin. Combining these requirements, we have for $f(x) = e^x$ that

$$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0)$$

This yields the approximation

$$p(x) = 1 + x + \tfrac{1}{2}x^2$$



y
y = e^x
y = p_1(x)
y = p_2(x)
2
1
-1
1
x

We continue this pattern, looking for a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

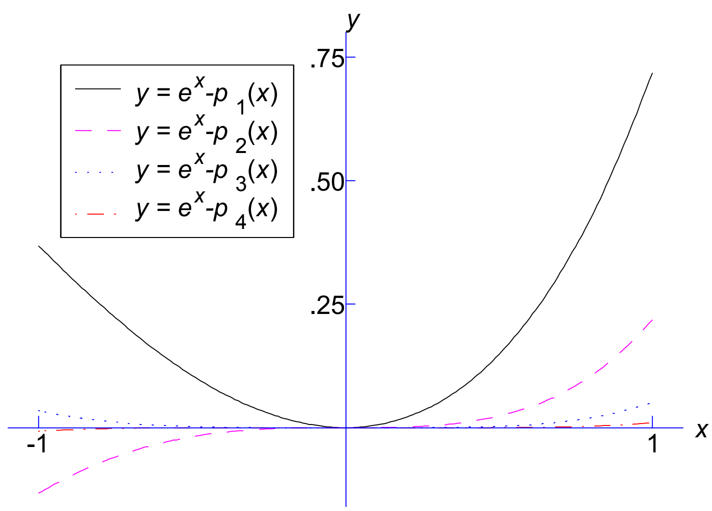
We now require that

$$p(0) = f(0), \quad p'(0) = f'(0), \quad \cdots, p^{(n)}(0) = f^{(n)}(0)$$

This leads to the formula

$$p(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$$

What are the problems when evaluating points $x$ that are far from 0?


y
.75
.50
.25
x
-1
1
y = e^x - p_1(x)
y = e^x - p_2(x)
y = e^x - p_3(x)
y = e^x - p_4(x)

# TAYLOR'S APPROXIMATION FORMULA

Let $f(x)$ be a given function, and assume it has derivatives around some point $x = a$ (with as many derivatives as we find necessary). We seek a polynomial $p(x)$ of degree at most $n$, for some non-negative integer $n$, which will approximate $f(x)$ by satisfying the following conditions:

$$\begin{aligned} p(a) &= f(a) \\ p'(a) &= f'(a) \\ p''(a) &= f''(a) \\ &\vdots \\ p^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

The general formula for this polynomial is

$$\begin{aligned} p_n(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) \\ + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) \end{aligned}$$

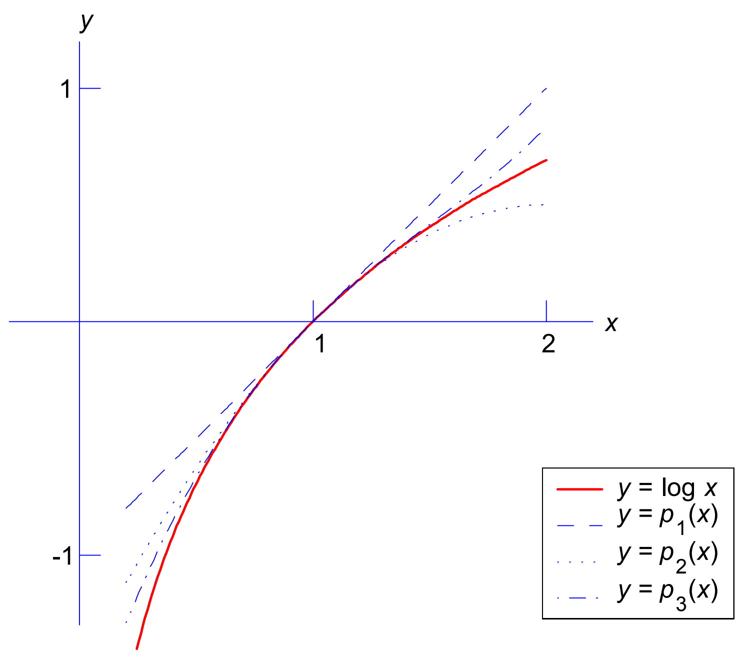
Then $f(x) \approx p_n(x)$ for $x$ close to $a$.

# TAYLOR POLYNOMIALS FOR $f(x) = \log x$

In this case, we expand about the point $x = 1$, making the polynomial tangent to the graph of $f(x) = \log x$ at the point $x = 1$. For a general degree $n \geq 1$, this results in the polynomial

$$\begin{aligned} p_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ + \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n \end{aligned}$$

Note the graphs of these polynomials for varying $n$.


y
1
-1
1
2
x
y = log x
y = p1(x)
y = p2(x)
y = p3(x)