# MTH 654

# Numerical Methods for Inverse Problems

## Homework 1

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Due: Oct 22

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1. Do [V] 1.12.

2. Confirm that the normal equations

$$A^T A\mathbf{f} = A^T\mathbf{d}$$

are equivalent to the least squares problem of minimizing

$$D(\mathbf{f}) = \|\mathbf{d} - A\mathbf{f}\|_2^2$$

by showing that

$$\frac{\partial D}{\partial f_i} = 0$$

can be written

$$\sum_{\ell=1}^{n}\left(\sum_{k=1}^{m} a_{ki}a_{k\ell}\right) f_\ell = \sum_{k=1}^{m} a_{ki}d_k.$$

3. Confirm that minimizing the Tikhonov functional

$$T_\alpha(\mathbf{f}) = \|\mathbf{d} - A\mathbf{f}\|_2^2 + \alpha\|\mathbf{f}\|_2^2 = \| \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} \mathbf{f} - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \|_2^2$$

is equivalent to solving the so-called *regularized normal equations*

$$\left(A^T A + \alpha I\right)\mathbf{f} = A^T\mathbf{d}.$$

4. Filtering can be described as replacing $\Sigma^+$ with $\Sigma_\alpha^+$ in $\mathbf{f} = V\Sigma^+U^T\mathbf{d}$, where

$$\Sigma^+ = \mathrm{diag}(\sigma_k^+)$$

with

$$\sigma_k^+ = \begin{cases} \frac{1}{\sigma_k} & \text{for } \sigma_k \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Give an expression for $\Sigma_\alpha^+$ in the case of *Truncated SVD*, i.e., when $w_\alpha(s^2) = H_\alpha(s^2)$.

(b) Give an expression for $\Sigma_\alpha^+$ in the case of $w_\alpha(s^2) = \max\{1, \frac{s^2}{\alpha}\}$.

5. Sketch the two filter functions described in 4 on a *semilogx* plot (similar to Figure 1.3). Which method is more computationally intensive?

6. Verify that the two filter functions described in 4 each satisfy equation (1.21), and therefore the method (with (1.23)) is convergent. (Note that bound for the latter filter function is tight.)

7. Verify that the filter function described in 4b defines a regularization method that satisfies

$$\|\mathbf{e}_\alpha^{\text{trunc}}\|_2^2 \leq \frac{4\alpha}{27}\|\mathbf{z}\|_2^2,$$

where $\mathbf{z}$ is defined in (1.25). Hint: see (1.26) and Exercise 1.7. This is a more restrictive bound than that for TSVD.