EVALUATING A POLYNOMIAL

Consider having a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

which you need to evaluate for many values of x. How do you evaluate it? This may seem a strange question, but the answer is not as obvious as you might think.

The standard way, written in a loose algorithmic format:

$$\begin{array}{l} poly = a_0 \\ for\ j = 1 : n \\ \quad poly = poly + a_j x^j \\ end \end{array}$$

To compare the costs of different numerical methods, we do an operations count, and then we compare these for the competing methods. Above, the counts are as follows:

$$additions: \quad n$$

$$multiplications: \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

This assumes each term $a_j x^j$ is computed independently of the remaining terms in the polynomial.

Next, do the terms x^j recursively:

$$x^j = x \cdot x^{j-1}$$

Then to compute $\left\{x^2, x^3, ..., x^n\right\}$ will cost $n-1$ multiplications. Our algorithm becomes

$$\begin{array}{l} poly = a_0 + a_1 x \\ power = x \\ for\ j = 2 : n \\ \quad power = x \cdot power \\ \quad poly = poly + a_j \cdot power \\ end \end{array}$$

The total operations cost is

$$\begin{aligned} additions:& \quad n \\ multiplications:& \quad n + n - 1 = 2n - 1 \end{aligned}$$

When n is evenly moderately large, this is much less than for the first method of evaluating $p(x)$. For example, with $n = 20$, the first method has 210 multiplications, whereas the second has 39 multiplications.

We now considered nested multiplication. As examples of particular degrees, write

$$
\begin{aligned}
n = 2: &\quad p(x) = a_0 + x(a_1 + a_2 x) \\
n = 3: &\quad p(x) = a_0 + x\,(a_1 + x\,(a_2 + a_3 x)) \\
n = 4: &\quad p(x) = a_0 + x\,(a_1 + x\,(a_2 + x\,(a_3 + a_4 x)))
\end{aligned}
$$

These contain, respectively, 2, 3, and 4 multiplications. This is less than the preceding method, which would have need 3, 5, and 7 multiplications, respectively.

For the general case, write

$$p(x) = a_0 + x\,(a_1 + x\,(a_2 + \cdots + x\,(a_{n-1} + a_n x) \cdots))$$

This requires n multiplications, which is only about half that for the preceding method. For an algorithm, write

$$
\begin{aligned}
&poly = a_n \\
&for\ j = n-1 : -1 : 0 \\
&\quad poly = a_j + x \cdot poly \\
&end
\end{aligned}
$$

With all three methods, the number of additions is n; but the number of multiplications can be dramatically different for large values of n.

NESTED MULTIPLICATION

Imagine we are evaluating the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

at a point $x = z$. Thus with nested multiplication

$$p(z) = a_0 + z\,(a_1 + z\,(a_2 + \cdots + z\,(a_{n-1} + a_n z) \cdots))$$

We can write this as the following sequence of operations:

$$\begin{aligned} b_n &= a_n \\ b_{n-1} &= a_{n-1} + z b_n \\ b_{n-2} &= a_{n-2} + z b_{n-1} \\ &\vdots \\ b_0 &= a_0 + z b_1 \end{aligned}$$

The quantities $b_{n-1}, ..., b_0$ are simply the quantities in parentheses, starting from the inner most and working outward.

Introduce

$$q(x) = b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}$$

Claim:

$$p(x) = b_0 + (x - z) q(x) \qquad (*)$$

Proof: Simply expand

$$b_0 + (x - z)\left(b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}\right)$$

and use the fact that

$$z b_j = b_{j-1} - a_{j-1}, \qquad j = 1, ..., n$$

With this result (*), we have

$$\frac{p(x)}{x - z} = \frac{b_0}{x - z} + q(x)$$

Thus $q(x)$ is the quotient when dividing $p(x)$ by $x - z$, and b_0 is the remainder.

If z is a zero of $p(x)$, then $b_0 = 0$; and then

$$p(x) = (x - z)q(x)$$

For the remaining roots of $p(x)$, we can concentrate on finding those of $q(x)$. In rootfinding for polynomials, this process of reducing the size of the problem is called deflation.

Another consequence of (*) is the following. Form the derivative of (*) with respect to x, obtaining

$$\begin{aligned} p'(x) &= (x - z)q'(x) + q(x) \\ p'(z) &= q(z) \end{aligned}$$

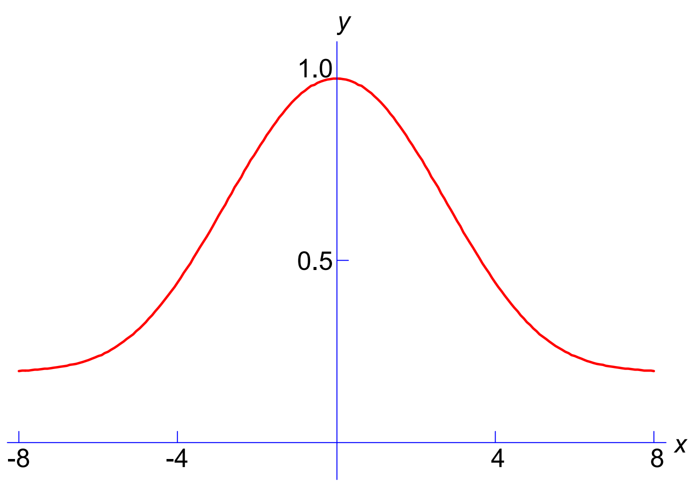
Thus to evaluate $p(x)$ and $p'(x)$ simultaneously at $x = z$, we can use nested multiplication for $p(z)$ and we can use the intermediate steps of this to also evaluate $p'(z)$. This is useful when doing rootfinding problems for polynomials by means of Newton's method.

APPROXIMATING $SF(x)$

Define

$$SF(x) = \frac{1}{x}\int_0^x \frac{\sin t}{t}dt, \qquad x \neq 0$$

We use Taylor polynomials to approximate this function, to obtain a way to compute it with accuracy and simplicity.



As an example, begin with the degree 3 Taylor approximation to $\sin t$, expanded about $t = 0$:

$$\sin t = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 \cos c_t$$

with c_t between 0 and t. Then

$$\begin{aligned}
\frac{\sin t}{t} &= 1 - \frac{1}{6}t^2 + \frac{1}{120}t^4 \cos c_t \\
\int_0^x \frac{\sin t}{t} dt &= \int_0^x \left[1 - \frac{1}{6}t^2 + \frac{1}{120}t^4 \cos c_t\right] dt \\
&= x - \frac{1}{18}x^3 + \frac{1}{120}\int_0^x t^4 \cos c_t dt \\
\frac{1}{x}\int_0^x \frac{\sin t}{t} dt &= 1 - \frac{1}{18}x^2 + R_2(x)
\end{aligned}$$

$$R_2(x) = \frac{1}{120}\frac{1}{x}\int_0^x t^4 \cos c_t \, dt$$

How large is the error in the approximation

$$SF(x) \approx 1 - \frac{1}{18}x^2$$

on the interval $[-1, 1]$? Since $|\cos c_t| \leq 1$, we have for $x > 0$ that

$$\begin{aligned} 0 \leq R_2(x) &\leq \frac{1}{120}\frac{1}{x}\int_0^x t^4 dt \\ &= \frac{1}{600}x^4 \end{aligned}$$

and the same result can be shown for $x < 0$. Then for $|x| \leq 1$, we have

$$0 \leq R_2(x) \leq \frac{1}{600}$$

To obtain a more accurate approximation, we can proceed exactly as above, but simply use a higher degree approximation to $\sin t$.

In the book we consider finding a Taylor polynomial approximation to $SF(x)$ with its error satisfying

$$|R_8(x)| \leq 5 \times 10^{-9}, \qquad |x| \leq 1$$

A Matlab program, `plot_sint.m`, implementing this approximation is given in the text and in the class account. The one in the class account includes the needed additional functions `sint_tay.m` and `poly_even.m`.

Begin with a Taylor series for $\sin t$,

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots + (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!} + (-1)^n \frac{t^{2n+1}}{(2n+1)!}\cos(c_t)$$

with c_t between 0 and t. Then write

$$\begin{aligned}\operatorname{Sint} x &= \frac{1}{x}\int_0^x \left[1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots + (-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}\right] dt + R_{2n-2}(x) \\ &= 1 - \frac{x^2}{3!3} + \frac{x^4}{5!5} - \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-1)!(2n-1)} + R_{2n-2}(x)\end{aligned}$$

$$R_{2n-2}(x) = \frac{1}{x}\int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!}\cos(c_t)\, dt$$

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To simplify matters, let $x > 0$. Since $|\cos(c_t)| \le 1$,

$$|R_{2n-2}(x)| \le \frac{1}{x}\int_0^x \frac{t^{2n}}{(2n+1)!}\,dt = \frac{x^{2n}}{(2n+1)!(2n+1)}$$

It is easy to see that this bound is also valid for $x < 0$. As required, choose the degree so that

$$|R_{2n-2}(x)| \le 5 \times 10^{-9}$$

From the error bound,

$$\max_{|x|\le 1} |R_{2n-2}(x)| \le \frac{1}{(2n+1)!(2n+1)}$$

Choose n so that this upper bound is itself bounded by 5×10^{-9}. This is true if $2n + 1 \ge 11$, i.e. $n \ge 5$.

The polynomial is

$$p(x) = 1 - \frac{x^2}{3!3} + \frac{x^4}{5!5} - \frac{x^6}{7!7} + \frac{x^8}{9!9}, \qquad -1 \le x \le 1$$

and

$$|SF(x) - p(x)| \le 5 \times 10^{-9}, \qquad |x| \le 1$$

To evaluate it efficiently, we set $u = x^2$ and evaluate

$$g(u) = 1 - \frac{u}{18} + \frac{u^2}{600} - \frac{u^3}{35280} + \frac{u^4}{3265920}$$

After the evaluation of the coefficients (done once), the total number of arithmetic evaluations is 4 additions and 5 multiplications to evaluate $p(x)$ for each value of x.