MTH 453/553 – Homework 4 Solutions

1. Consider the initial value problem

$$u_t - u_x = 0,$$
$$u(x,0) = \sin(\omega x),$$

where ω is a parameter. The exact solution to this problem is

$$u(x,t) = \sin(\omega(t+x)).$$

Let us solve this problem on the interval $[-\pi, \pi]$ with periodic boundary conditions, i.e.,

$$u(t,-\pi) = u(t,\pi),\ t > 0.$$

We perform a discretization of the spatial interval $[-\pi, \pi]$ with spatial step size h and we use a temporal step size k on the time interval [0,1]. Let $m+1 = \frac{2\pi}{h}$. We want to calculate approximate solution values u_j^n for $j = 1, \ldots, m+1$. The periodic boundary conditions give

$$u_0^n = u_{m+1}^n,\ \forall n$$

It is helpful to imagine the spatial mesh points sitting on a ring, rather than on a straight line, with x_{m+1} being identified with x_0. Thus, when $j = m$, then $u_{j+1} = u_0$ and for $j = 0, u_{j-1} = u_m$. (This gets further complicated in MATLAB as zero index is not allowed, so we add one to each index.)

(a) Download the code `leapfrog.m` from the course webpage. This code implements the Leapfrog (midpoint) method (CC), given in (10.13) in the text, to obtain a numerical solution to the given problem.

(b) Modify this code to create a new file that implements the forward in time-backward in space (FB) finite difference scheme given in (10.23) to solve the given problem.

(c) Create a new file that implements the forward in time-centered in space (FC) finite difference scheme given in (10.5) to solve the given problem.

(d) Calculate the maximum errors at $t = 1$ for all three schemes using the values of ω, m and ν $(= k/h)$ given in the table below and fill in the missing entries in the table

ω	$m+1$	ν	Error in FB	Error in FC	Error in CC
2	20	0.5	0.2473	0.3304	0.0788
2	200	0.5	?	?	?
2	2000	0.5	?	?	?

What order of accuracy does each method seem to exhibit?

Answer: Download the Codes

i. `downwind.m`

ii. `forward_center.m`

iii. `leapfrog.m`

from my webpage. Run these codes with the script `hw4prob1d.m` in MATLAB to see the table of errors. You should see 1st order, unstable, 2nd order, respectively.

(e) Fix $m+1 = 20$. What is the effect on the error of each method when ω is increased or decreased? **Answer:** Run the above codes with the script `hw4prob1e.m` in MATLAB to see the table of errors. You should see that for $\omega < 1$ periodic boundary conditions do not make sense, and for $\omega > 1$ error generally gets worse as the frequency increases. Downwind scheme with $\omega = 8$ is particularly interesting.

(f) Fix $m+1 = 20$. What is the effect on the error of each method when ν is increased or decreased? In particular, try $\nu = 1$ and $\nu > 1$. **Answer:** Run the above codes with the script `hw4prob1f.m` in MATLAB to see the table of errors. You should see that for $\nu = 1$ downwind and leapfrog are exact. The FC method seems to be best for very small ν. Downwind gets progressively worse for $\nu > 1$. However, something interesting happens to downwind and FC for $\nu > \pi$. Also see leapfrog for $\pi/2 < \nu < \pi$.

2. Download the code `leapfrogpulse.m` from the course webpage. This code implements the Leapfrog method with a Gaussian pulse for the initial condition. Snapshots of the solution at various times are shown, and the norm of the solution is displayed to the prompt.

(a) Run the code `leapfrogpulse.m` with $m+1 = 75$ and try ν values of 1, 0.8, and 1.01. Describe the resulting qualitative behavior of solutions and explain why it occurs. **Answer:** Hopefully you observed an exact solution, a dispersive solution and an unstable solution, respectively.

(b) Change the boundary condition from periodic to $u(\pi, t) = 0$. Impose *artificial absorbing boundary conditions* at the outflow boundary $x = -\pi$:

$$U_1^{n+1} = U_1^n + \nu\,(U_2^n - U_1^n)$$

Run the code with $m+1 = 75$ and try ν values of 1, 0.8, and 1.01. (Try also the case $\nu = 0.25$, and look closely for a "reflection" that travels backwards!) Discuss the accuracy of the absorbing boundary condition in terms of the remaining "energy". **Answer:** Download the code `leapfrogpulsekey.m`.

3. Consider the advection equation

$$u_t + au_x = 0,$$

and recall that the forward in time-centered in space (FC) scheme is consistent but unconditionally unstable. The *Lax-Friedrichs* scheme is a variation of the FC scheme and is given as

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\nu}{2}\left(U_{j+1}^n - U_{j-1}^n\right),$$

where $\nu = \frac{ak}{h}$ is the *Courant number.*

(a) (**553:**) Show that the Lax-Friedrichs scheme is consistent and find the order of accuracy.

Answer: Compute the local truncation error. **Note**: To do this, you need to write the difference scheme in a form that directly models the derivatives and then substitute the exact solution in it. Rewrite the difference scheme as

$$\frac{U_j^{n+1} - \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right)}{k} + a\frac{\left(U_{j+1}^n - U_{j-1}^n\right)}{2h} = 0$$

Substituting the exact solution in the above, the local truncation error at (x_j, t_n) is given as

$$\tau_j^n = \frac{\left\{u(x_j, t_{n+1}) - \dfrac{(u(x_{j-1}, t_n) + u(x_{j+1}, t_n))}{2}\right\}}{k} + a\frac{\{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\}}{2h} \tag{1}$$

Using Taylor series expansions around (x_j, t_n) we have

$$u(x_j, t_{n+1}) = u(x_j, t_n) + ku_t(x_j, t_n) + \frac{k^2}{2}u_{tt}(x_j, t_n) + O(k^3) \tag{2}$$

$$u(x_{j+1}, t_n) = u(x_j, t_n) + hu_x(x_j, t_n) + \frac{h^2}{2}u_{xx}(x_j, t_n) + O(h^3) \tag{3}$$

$$u(x_{j-1}, t_n) = u(x_j, t_n) - hu_x(x_j, t_n) + \frac{h^2}{2}u_{xx}(x_j, t_n) + O(h^3) \tag{4}$$

Substituing (2), (3) and (4) in (1) we get

$$\begin{aligned} \tau_j^n = &\, u_t(x_j, t_n) + \frac{k}{2}u_{tt}(x_j, t_n) + O(k^2) \\ &- \frac{h^2}{2k}u_{xx}(x_j, t_n) + \frac{O(h^3)}{k} \\ &+ au_x(x_j, t_n) + O(h^2) \end{aligned}$$

Since, u is the exact solution, we have $u_t + au_x = 0$. Also letting the Courant number $\nu = ak/h$ be constant, then $h/k = a/\nu$ and we have

$$\tau_j^n = \frac{k}{2}u_{tt}(x_j, t_n) - \frac{ah}{2\nu}u_{xx}(x_j, t_n) + O(k^2) + O(h^2) = O(k) + O(h) \tag{5}$$

Thus, the scheme is consistent and it is first order accurate in both space and time.

(b) Use von Neumann analysis to show that the Lax-Friedrichs scheme is stable, provided that the CFL condition, $|\nu| \leq 1$, holds. **Answer:** Applying the Fourier transform to the difference scheme we have

$$\hat{U}(\xi, t_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-ix\xi} U(x, t_n) dx \tag{6}$$

we have the equation

$$\hat{U}(\xi, t_{n+1}) = \left\{ \frac{1}{2} \left(\mathrm{e}^{-i\xi h} + \mathrm{e}^{i\xi h} \right) - \frac{\nu}{2} \left(\mathrm{e}^{i\xi h} - \mathrm{e}^{-i\xi h} \right) \right\} \hat{U}(\xi, t_n)$$

Thus, the amplification factor is

$$\begin{aligned} g(h\xi) &= \frac{1}{2} \left(\mathrm{e}^{-i\xi h} + \mathrm{e}^{i\xi h} \right) - \frac{\nu}{2} \left(\mathrm{e}^{i\xi h} - \mathrm{e}^{-i\xi h} \right) \\ &= \left(\cos(h\xi) - \mathrm{i}\nu \sin(h\xi) \right) \end{aligned}$$

Hence

$$|g(h\xi)| = \sqrt{\cos^2(h\xi) + \nu^2 \sin^2(h\xi)}$$

Thus, if $|\nu| \leq 1$, then we have $|g(h\xi)| \leq 1$, for any ξ and hence the scheme is stable.