MTH 453/553 – Homework 4

1. Consider the initial value problem

$$u_t - u_x = 0,$$
$$u(x, 0) = \sin(\omega x),$$

where ω is a parameter. The exact solution to this problem is

$$u(x, t) = \sin(\omega(t + x)).$$

Let us solve this problem on the interval $[-\pi, \pi]$ with periodic boundary conditions, i.e.,

$$u(t, -\pi) = u(t, \pi),\ t > 0.$$

We perform a discretization of the spatial interval $[-\pi, \pi]$ with spatial step size h and we use a temporal step size k on the time interval [0,1]. Let $m + 1 = \frac{2\pi}{h}$. We want to calculate approximate solution values u_j^n for $j = 1, \ldots, m + 1$. The periodic boundary conditions give

$$u_0^n = u_{m+1}^n,\ \forall n$$

It is helpful to imagine the spatial mesh points sitting on a ring, rather than on a straight line, with x_{m+1} being identified with x_0. Thus, when $j = m$, then $u_{j+1} = u_0$ and for $j = 0, u_{j-1} = u_m$. (This gets further complicated in MATLAB as zero index is not allowed, so we add one to each index.)

(a) Download the code `leapfrog.m` from the course webpage. This code implements the Leapfrog (midpoint) method (CC), given in (10.13) in the text, to obtain a numerical solution to the given problem.

(b) Modify this code to create a new file that implements the forward in time-backward in space (FB) finite difference scheme given in (10.23) to solve the given problem.

(c) Create a new file that implements the forward in time-centered in space (FC) finite difference scheme given in (10.5) to solve the given problem.

(d) Calculate the maximum errors at $t = 1$ for all three schemes using the values of ω, m and ν $(= k/h)$ given in the table below and fill in the missing entries in the table

ω	$m + 1$	ν	Error in FB	Error in FC	Error in CC
2	20	0.5	0.2473	0.3304	0.0788
2	200	0.5	?	?	?
2	2000	0.5	?	?	?

What order of accuracy does each method seem to exhibit?

(e) Fix $m+1=20$. What is the effect on the error of each method when ω is increased or decreased?

(f) Fix $m+1=20$. What is the effect on the error of each method when ν is increased or decreased? In particular, try $\nu=1$ and $\nu>1$.

2. Download the code `leapfrogpulse.m` from the course webpage. This code implements the Leapfrog method with a Gaussian pulse for the initial condition. Snapshots of the solution at various times are shown, and the norm of the solution is displayed to the prompt.

   (a) Run the code `leapfrogpulse.m` with $m+1=75$ and try ν values of 1, 0.8, and 1.01. Describe the resulting qualitative behavior of solutions and explain why it occurs.

   (b) Change the boundary condition from periodic to $u(\pi,t)=0$. Impose *artificial absorbing boundary conditions* at the outflow boundary $x=-\pi$:

$$U_1^{n+1} = U_1^n + \nu\left(U_2^n - U_1^n\right)$$

   Run the code with $m+1=75$ and try ν values of 1, 0.8, and 1.01. (Try also the case $\nu=0.25$, and look closely for a "reflection" that travels backwards!) Discuss the accuracy of the absorbing boundary condition in terms of the remaining "energy".

3. Consider the advection equation

$$u_t + a u_x = 0,$$

and recall that the forward in time-centered in space (FC) scheme is consistent but unconditionally unstable. The *Lax-Friedrichs* scheme is a variation of the FC scheme and is given as

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\nu}{2}\left(U_{j+1}^n - U_{j-1}^n\right),$$

where $\nu = \frac{ak}{h}$ is the *Courant number.*

   (a) (**553:**) Show that the Lax-Friedrichs scheme is consistent and find the order of accuracy.

   (b) Use von Neumann analysis to show that the Lax-Friedrichs scheme is stable, provided that the CFL condition, $|\nu| \le 1$, holds.