MTH 453/553 – Homework 3 Solutions

1. Download the Codes

   (a) `ExplicitHE.m`

   (b) `ImplicitHE.m`

   (c) `CrankHE.m`

   from my webpage. Run these codes with the required values of $h = dx$ and $k = dt$ in MATLAB.

2. Consider the following nonlinear heat equation

$$\begin{aligned} u_t &= (\alpha(u)u_x)_x, \text{ for } x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x), \end{aligned}$$

   where $\alpha(u)$ is a given strictly positive and smooth function.

   (a) Using Taylor series expansions around (x,t) we have

$$\frac{u(x,t+k) - u(x,t)}{k} = u_t(x,t) + \frac{k}{2}u_{tt}(x,t) + O(k^2)$$

   Thus,

$$\frac{u(x,t+k) - u(x,t)}{k} = u_t(x,t) + O(k) \tag{1}$$

   Let $v = \alpha(u)u_x$. Using Taylor series expansions around (x,t) we have

$$\frac{v(x+h/2,t) - v(x-h/2,t)}{h} = v_x(x,t) + \frac{h^2}{48}v_{xxx}(x,t) + O(h^4)$$

   Thus,

$$\frac{v(x+h/2,t) - v(x-h/2,t)}{h} = v_x(x,t) + O(h^2) \tag{2}$$

   (b) Using Taylor Series expansions around the point $c^* = (x+h/2,t)$ we have

$$u(x+h,t) = u(c^*) + \frac{h}{2}u_x(c^*) + \frac{h^2}{8}u_{xx}(c^*) + O(h^3) \tag{3}$$

$$u(x,t) = u(c^*) - \frac{h}{2}u_x(c^*) + \frac{h^2}{8}u_{xx}(c^*) + O(h^3) \tag{4}$$

Subtracting the two equations above, we get

$$\frac{u(x+h,t)-u(x,t)}{h} = u_x(c^*) + O(h^2) \tag{5}$$

Let $u^* = u(c^*)$. Then, we have

$$\alpha(u(x+h,t)) = \alpha(u^*) + \frac{h}{2}\alpha_x(u^*) + \frac{h^2}{8}\alpha_{xx}(u^*) + O(h^3) \tag{6}$$

$$\alpha(u(x,t)) = \alpha(u^*) - \frac{h}{2}\alpha_x(u^*) + \frac{h^2}{8}\alpha_{xx}(u^*) + O(h^3) \tag{7}$$

Adding the two equations above and dividing by half, gives

$$\frac{\alpha(u(x+h,t)) + \alpha(u(x,t))}{2} = \alpha(u^*) + \frac{h^2}{4}\alpha_{xx}(u^*) + O(h^4) \tag{8}$$

Thus,

$$\frac{\alpha(u(x+h,t)) + \alpha(u(x,t))}{2} = \alpha(u^*) + O(h^2) \tag{9}$$

Using the approximations in (5) and (9) above, we get

$$\begin{aligned} v(c^*) &= \alpha(u^*)u(c^*) \\ &= \left(\frac{\alpha(u(x+h,t)) + \alpha(u(x,t))}{2} + O(h^2)\right)\left(\frac{u(x+h,t)-u(x,t)}{h} + O(h^2)\right) \end{aligned}$$

Thus,

$$v(c^*) = \left(\frac{\alpha(u(x+h,t)) + \alpha(u(x,t))}{2})\right)\left(\frac{u(x+h,t)-u(x,t)}{h}\right) + O(h^2) \tag{10}$$

(c) Using approximations (1), (2) and (10) above we can derive the scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\alpha_{j+1/2}^n(U_{j+1}^n - U_j^n) - \alpha_{j-1/2}^n(U_j^n - U_{j-1}^n)}{h^2}, \tag{11}$$

where $\alpha_{j+1/2}^n = (\alpha(U_{j+1}^n) + \alpha(U_j^n))/2$.

3. Consider the nonlinear heat equation

$$\begin{aligned} u_t &= (uu_x)_x, \text{ for } x \in (0,1), 0 < t \leq 1 \\ u(0,t) &= t, \; u(1,t) = 1+t \\ u(x,0) &= x \end{aligned}$$

(b) Show, by induction, that the explicit scheme in Problem 2, part c, gives the exact solution at each grid point, i.e., show that $U_j^n = x_j + t_n$ for any grid sizes.

Solution: When $n = 0$ we have $U_j^0 = x_j$, so that the scheme has the exact solution at each grid point initially.

Assume that the explicit solution gives the exact solution at each grid point at time $t_n = nk$, i.e., $U_j^n = x_j + t_n$. From the scheme, we have

$$U_j^{n+1} = U_j^n + k\frac{\alpha_{j+1/2}^n(U_{j+1}^n - U_j^n) - \alpha_{j-1/2}^n(U_j^n - U_{j-1}^n)}{h^2}, \tag{12}$$

We note that,

$$U_{j+1}^n - U_j^n = (x_j + h + t_n) - (x_j + t_n) = h \tag{13}$$

$$U_j^n - U_{j-1}^n = (x_j + t_n) - (x_j - h + t_n) = h \tag{14}$$

$$\begin{aligned}\alpha_{j+1/2}^n &= \frac{\alpha(U_{j+1}^n) + \alpha(U_j^n)}{2} = \frac{U_{j+1}^n + U_j^n}{2} = \frac{(x_j + h + t_n) + (x_j + t_n)}{2} \\ &= (x_j + t_n) + h/2\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}\alpha_{j-1/2}^n &= \frac{\alpha(U_j^n) + \alpha(U_{j-1}^n)}{2} = \frac{U_j^n + U_{j-1}^n}{2} = \frac{(x_j + t_n) + (x_j - h + t_n)}{2} \\ &= (x_j + t_n) - h/2\end{aligned} \tag{16}$$

Substituting (13)-(16) in (12) we get

$$U_j^{n+1} = x_j + t_n + k\frac{h(x_j + t_n + h/2) - h(x_j + t_n - h/2)}{h^2} \tag{17}$$

$$= x_j + t_n + k = x_j + t_{n+1} \tag{18}$$

Since, j is a generic index, using induction we have shown that the explicit scheme gives the exact solution at each grid point.

(d) From the numerical results obtained in part c, it is clear that some kind of stability condition is needed. Try to come up with a stability condition for this problem. Run some numerical experiments with mesh parameters satisfying this condition. Are the numerical solutions well-behaved if the conditions on the mesh parameters are satisfied?

Solution: We linearize the equation and freeze the coefficients by considering the problem locally. This leads to a linear problem with constant coefficients. For this linear problem, von Neumann analysis can be applied, and a stability condition can be derived. This condition will depend on the frozen coefficients involved. The trick is then to choose a conservative time step, covering all possible values of the frozen coefficient.

Consider the given problem locally, i.e., close to some fixed location (x_0, t_0). If u is smooth, we can approximate it by a constant value

$$\alpha_0 = u(x_0, t_0)$$

close to the point (x_0, t_0). This approximation leads to the equation

$$u_t = \alpha_0 u_{xx},$$

and the associated scheme,

$$\frac{U_j^{n+1} - U_j^n}{k} = \alpha_0 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \tag{19}$$

Applying von Neumann analysis to this linear problem, (we have seen this in class) we come up with the stability condition

$$\alpha_0 k/h^2 \leq 1/2$$

Thus, for mesh parameters satisfying this bound, the scheme is stable, at least locally. In order to derive a global bound, we observe that the exact solution

$$u(x,t) = x + t \leq 2$$

for $x \in [0,1]$ and $0 \leq t \leq 1$. Thus, any frozen coefficient α_0 is less than or equal to 2, and thus the most restrictive requirement on the time step is given by

$$k \leq \frac{h^2}{4}$$